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ON ORDER AND MORPHISMS RELATED TO A SHEFFER STROKE

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ABSTRACT

The paper deals with a new interpretation of a special functional characterisation of Sheffer strokes, with the study of morphisms and the construction of different De Morgan Algebras on a given set.

Introduction.

The study of De Morgan lattices through one binary connective S, known as "Sheffer Stroke", was introduced in [Monteiro and Picco, 1963] and recently in [Trillas and Alsina, 1981] a functional characterisation was presented. In the present paper the study of this functional characterisation is extended, establishing the relationship of the Sheffer Stroke (Ss) itself with the underlying ordered structure and the meaning of the functional characterisation in this context. Furthermore, the study of morphisms of Ss enable us to construct new De Morgan Algebras on the same set M; one of the two possible constructions changes the lattice-structure, but not the complement, while the other changes both, but allowing to construct a whole family of morphisms generating a family of De Morgan lattices with the same lattice struc ture and different complements. Proofs are omitted as far as they are straightforward and tedious computations.

By definition [Monteiro and Picco, 1963] a <u>Ss</u> on a non-empty set M is a mapping S from MxM into M such that

(a) $S(S(x,x),S(x,y)) = x$,
(b) $S(S(x,S(y,z)),S(x,S(y,z)))=S(S(S(y,y),x),S(S(z,z),x))$,

for any x,y,z in M.

The functional characterization presented in [Trillas and Alsina, 1981] is expressed by the following

<u>Theorem 1.</u> A function S from MxM into M is a <u>Ss</u> if and only if

$$S = f\cdot(n\times n),$$

where n is an involutive function on M, and f is a function from MxM into M satisfying

(a) $f(x,n(f(n(x),n(y)))) = x$,
(b) $f(x,n(f(n(y),n(z)))) = n(f(n(f(y,x)),n(f(z,x))))$,

for any x,y,z in M.

Notice that this is a reduced version, because the original one included the idempotency of f ($f(x,x)=x$ for any x in M); this condition has been dropped as far as it can be deduced directly from the other two conditions (a) and (b).

From this theorem the main result of [Monteiro and Picco, 1963] can easily be deduced and can be stated as

<u>Corollary 2.</u> If S is a <u>Ss</u> on M, defining

$$\neg x = S(x,x),\quad x\wedge y = S(\neg x,\neg y),\quad x\vee y = \neg S(x,y),$$

$(M,\neg,\wedge,\vee)$ is a De Morgan Lattice.

Conversely, any De Morgan Lattice induces a <u>Ss</u> given by $S(x,y) = \neg x \wedge \neg y$, which suggests to read $S(x,y)$ as "neither x nor y".

From the functional characterization given in theorem 1 the following result is directly deduced

<u>Corollary 3.</u> Given a <u>Ss</u> S both the involutive function n and the function f from theorem 1 associated both are unique.

The order induced by a Sheffer Stroke.

Let S be a <u>Ss</u> on a non-empty set M. Define in M the following binary relation

$$x \leqslant_s y \text{ iff } S(x,y) = S(y,y).$$

Then

<u>Theorem 4.</u> $(M, \leqslant_s)$ is a distributive lattice and the function n defined in M by $n(x) = S(x,x)$ is non-increasing and involutive. The order relation $\leqslant_s$ is a total order iff $S(x,y)=S(y,y)$ or $S(x,y)=S(x,x)$ for all x,y in M.

We will write $\inf_s(x,y)$ and $\sup_s(x,y)$ for the infimum and the supremum of the pair (x,y) relatively to the partial order $\leqslant_s$, called the order induced by the <u>Ss</u> S.

The converse of theorem 4 also holds: <u>If $(M,\leqslant)$ is a distributive lattice and n is a non increasing and involutive function on M, $S = \inf_{\leqslant}(nxn)$ is a Ss on M.</u>

Notice that defining $x \leqslant'_s y$ iff $S(x,y)=S(x,x)$ the dual order on M is obtained, and that if there exists a function h on M such that

$$x \leqslant_h y \text{ iff } S(x,y) = h(y),$$

is a partial order, then the reflexive property implies

$$h(x) = S(x,x) \text{ for any } x \text{ in } M.$$

Notice also that for the converse of theorem 4 we get a similar result considering the following alternative definition:

$$S' = \sup_{\leqslant}(n \times n).$$

We call S' the dual <u>Ss</u> of S.

The proof of theorem 4 can be obtained either by straighforward computation or considering the equivalence between the <u>Ss</u> and the De Morgan lattice structure on the set M stated in corollary 2. But it is interesting to see that the ordered structure can be directly derived from the binary connective S and reciprocally. Nonwithstanding the most important result lies in the direction of the functional characterization, for we obtain the following

<u>Theorem 5.</u> The function f of theorem 1 is the infimum with respect to the order relation induced by S (respectively the supremum for the dual order).

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Consider now a <u>Ss</u> S on M, and a one-to-one and onto mapping h from a non-empty set N onto M. Let be $\leqslant_s$ the order induced by S on M and $n(x) = S(x,x)$ the induce involutive function.

<u>Theorem 6.</u> $S^* = h^{-1} \circ S \circ (h \times h)$ is a <u>Ss</u> on N such that

(a) the involutive function induced by S^* on N is $n^* = h^{-1} \circ n \circ h$;

(b) h is an order isomorphism between N with the order induced by S^* and $(M, \leqslant_s)$;

(c) n^* is non-decreasing with respect to the induced order in N.

Restricting our study to the case N = M, suppose now that $h \circ n \circ h^{-1}$ is non-increasing with respect to $\leqslant_s$. Under this hypothesis next theorem gives an alternative way of defining a Ss on M:

Theorem 7. $\bar{S} = h^{-1} \circ S \circ (n \circ h \circ n x n \circ h \circ n)$ is a Ss on M such that

(a) the induced involutive function is $\bar{S}(x,x) = n(x) = S(x,x)$.

(b) h is an order isomorphism between M with the order induced by $\bar{S}$ and $(M, \leqslant_s)$.

Considering both ways of defining new Ss on M for the same one-to-one and onto function h (supposed $h \circ n \circ h^{-1}$ non-increasing for $\leqslant_s$) it is easily verified, due to both properties (b) from theorems 6 and 7, that:

The order induced by S* and the order induced by $\bar{S}$ on M are the same.

Notice that if nh = hn, then $S^* = \bar{S}$, but if on the contrary nh ≠ hn, then new De Morgan lattice-structures on M are obtained which differ from the original one (corresponding to S) for S* both in the underlying lattice-structure and the complement (given by n*), and for $\bar{S}$ only in the underlying lattice-structure, resting the complement the same ($\bar{n} = n$).

Defining now a family of functions:

$$h_k = n \circ h \circ (n \circ h^{-1} \circ n \circ h)^{k-1} \circ n, \; k \text{ in } N,$$

being h a one-to-one and onto function M and n the involutive function considered before, then

$$S_k = h_k^{-1} \circ S \circ (h_k x h_k), \; k \text{ in } N,$$

is a family of Ss on M, due to theorem 6.

Denoting by $\leqslant_k$ the order induced in M by S_k and by n_k the corresponding involutive function, we have:

<u>For all k in N the order $\leq_k$ is the same, differing the involutive functions n_k iff $nh \neq hn$.</u>

To remark the close relation between the ordered structure and the <u>Ss</u> on M, let us state finally

<u>Theorem 8.</u> Consider the set M ordered both by $\leq_s$ (the order induced by S) and an arbitrary order α. Let be h an order isomorphism between (M,α) and $(M,\leq_s)$. Then

(a) $n' = h^{-1}\circ n\circ h$ is an involutive function on M, non-increasing for α;

(b) if h is increasing, $S^*=h^{-1}\circ S\circ(hxh)=\inf_\alpha(n^*xn^*)$, and if h is decreasing $S^*=\sup_\alpha(n^*xn^*)$.

(c) if n is non-increasing for α then $h\circ n\circ h^{-1}$ is non-increasing for $\leq_s$, and $\bar{S} = h^{-1}\circ S\circ(n\circ h\circ nxn\circ h\circ n)$ is a <u>Ss</u> on M such that the order induced by S is precisely α, both for h increasing and for h decreasing.

The fact of being h an order isomorphism allows to give a negative answer to an open problem stated in [Trillas and Alsina, 1981]. There the authors asked whether any total order in [0,1] could be described through a <u>Ss</u> defined by

$$S(x,y)=g^{-1}(\min(g(n(x)),g(n(y)))),$$

where g is a one-to-one function from [0,1] onto itself, n an involutive, decreasing function from [0,1] onto itself such that $n(0) = 1$, min the minimum with respect to the usual order on [0,1] and n and g verify that $g\circ n\circ g^{-1}$ is non-increasing.

The negative answer can be obtained, for example, as follows; Writing $\bar{S} = \min(nxn)$, then $S = g^{-1}\bar{S}(n\circ g\circ nxn\circ g\circ n)$ and theorem 7 implies that g is an order isomorphism between [0,1] with the usual order and [0,1] with the order induced by S. So it is enough to bring a total order on [0,1] not isomorphic to the usual one to disprove the statement. Consider therefore the following ordering of [0,1]: "the usual order in [0,1/2)∪(1/2,1], and x less than 1/2 for any x in [0,1/2)∪(1/2,1]". It is easy to prove that there exists no isomorphism, as required.

Allthough generally the answer is a negative one, the description of any total order isomophic to the usual order in [0,1] through a <u>Ss</u> is ensured by theorem 8.

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