STOCHASTICA Vol. VII, Nº 3, 1983

THE SPACE OF DISTRIBUTION FUNCTIONS IS SEPARABLE

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ABSTRACT

The space of distribution functions endowed with the metric introduced in [5] *is separable.*

The space Δ of distribution functions (d.f.'s) is a compact (and hence complete) metric space when it is endowed with the metric $d: \Delta\times\Delta \to R^+$ defined by

$$(1) \qquad d(F,G) := \sum_{r=1}^{\infty} 2^{-r} \left| \int_{\bar{R}} \theta_r \, dF - \int_{\bar{R}} \theta_r \, dG \right|$$

where $\bar{R} = R\cup\{-\infty,+\infty\}$ are the extended reals and $\{\theta_r, r \in N\}$ is a suitable enumeration of the functions $\phi_{ab}:\bar{R} \to [0,1]$ defined, for every pair $a,b \in Q$ with $a<b$, by

$$\phi_{ab}(x) := (b-a)^{-1} \int_x^{+\infty} 1_{[a,b]}(t)\, dt.$$

The convergence determined by this metric is the usual weak convergence of d.f.'s ([2]) which will be denoted by $F_n \xrightarrow{w} F$. For what precedes see [5].

Let $E:=\{\varepsilon_x\in\Delta:\ x\in\bar{R}\}$ where $\varepsilon_x:\bar{R}\to[0,1]$ is the d.f.

$$\varepsilon_x(t):=\begin{cases}0, & t<x,\\ 1, & t\geqslant x.\end{cases}$$

Theorem 1. E is homeomorphic to $\bar{R}$.

Proof. There is obviously a bijection $\Phi:\bar{R}\to E$ such that $\Phi(x)=\varepsilon_x$. One must now show that both Φ and Φ^{-1} are continuous.

(i) <u>Φ is continuous.</u> Let $\{x(n):n\in N\}\subset\bar{R}$ be a sequence that converges to $x\in\bar{R}$. Now $d(\varepsilon_{x(n)},\varepsilon_x)=\Sigma_r\ 2^{-r}\ \delta(r,n)$ where

$$\delta(r,n):=\left|\int_{\bar{R}}\theta_r\ d\varepsilon_{x(n)}-\int_{\bar{R}}\theta_r\ d\varepsilon_x\right|.$$

If $a,b\in Q$, then, by integration by parts

$$\left|\int_{\bar{R}}\phi_{ab}\ d\varepsilon_{x(n)}-\int_{\bar{R}}\phi_{ab}\ d\varepsilon_x\right|=(b-a)^{-1}\left|\int_a^b\{\varepsilon_{x(n)}(t)-\varepsilon_x(t)\}dt\right|\leqslant$$

$$\leqslant(b-a)^{-1}\int_a^b|\varepsilon_{x(n)}(t)-\varepsilon_x(t)|dt.$$

But

$$|\varepsilon_{x(n)}(t)-\varepsilon_x(t)|=\begin{cases}0, & t\notin[\min\{x,x(n)\},\max\{x,x(n)\}[\,,\\ 1, & t\in[\min\{x,x(n)\},\max\{x,x(n)\}[\,.\end{cases}$$

Therefore, by dominated convergence,

$$\lim_{n\to\infty}\int_{\bar{R}}\phi_{ab}\ d\varepsilon_{x(n)}=\int_{\bar{R}}\phi_{ab}\ d\varepsilon_x\qquad\forall\ a,b\in Q(a<b),$$

viz. $\lim_{n\to\infty}\delta(r,n)=0\ \forall r\in N$.

Thus, applying the dominated convergence theorem to the counting measure on N, one has $\lim_n d(\varepsilon_{x(n)},\varepsilon_x)=0$, i.e. $x(n)\to x$ implies $\varepsilon_{x(n)} \overset{w}{\to} \varepsilon_x$, which proves that Φ is continuous.

(ii) <u>ϕ^{-1} is continuous.</u> Assume $\varepsilon_{x(n)} \overset{w}{\to} \varepsilon_x$ i.e. $d(\varepsilon_{x(n)}, \varepsilon_x) \to 0$, with x, $x(n) \in \bar{R}$ $(n \in N)$. One ought to show that $x(n) \to x$ in the usual topology. $\varepsilon_{x(n)} \overset{w}{\to} \varepsilon_x$ means that

$$\lim_{n\to\infty} \;_{x(n)}(t) = \begin{cases} 0, & t<x, \\ 1, & t>x; \end{cases}$$

the limit is undefined at $t=x$.

If one assumes, <u>ab absurdo</u>, that $\{x(n) : n \in N\}$ does not converge to x, then one can find $\delta>0$ and an infinite subset J of N such that $|x(n)-x| \geqslant \delta$ or $x(n) \notin \,]x-\delta, x+\delta[$, whenever $n \in J$. Then, if $n \in J$, one has for all $a, b \in Q$ with $a \leqslant x-\delta$ and $b \geqslant x+\delta$

$$\left|\int_R \phi_{ab}\, d\varepsilon_{x(n)} - \int_R \phi_{ab}\, d\varepsilon_x\right| = (b-a)^{-1} \left| \int_R \{\varepsilon_{x(n)}(t) - \varepsilon_x(t)\} dt \right| \geqslant$$

$$\geqslant (b-a)^{-1} \lambda(\,]a,b[\, \cap \,]x-\delta, x+\delta[\,) = 2\delta/(b-a),$$

where λ is Lebesgue measure. Now, if $b \in [x+\delta, x+2\delta]$ and $a \in [x-2\delta, x-\delta]$, one has $2\delta/(b-a) > 1/2$ so that a denumerable subset K of N exists such that, if $n \in J$,

$$d(\varepsilon_{x(n)}, \varepsilon_x) \geqslant \sum_{r \in K} 2^{-r} \delta(r,n) \geqslant (1/2) \sum_{r \in K} 2^{-r} > 0$$

which contradicts the assumption. Therefore $x(n) \to x$ in the usual topology and ϕ^{-1} is continuous. Q.E.D.

<u>Theorem 2.</u> E is separable.

Proof. The subset $D := \{\varepsilon_q : q \in Q\} \subset E$ is denumerable and dense in E, by the previous result, and $\bar{Q} = \bar{R}$. Q.E.D.

The following result can now be proved in the same way as the classical one for $(\Delta_\circ, d_L)$ (see, e.g., [1] exercise 25.4): d_L

is the Lévy metric on Δ_o, the space of "proper" d.f.'s.

Theorem 3. For every $\delta>0$ and for every $F \in \Delta$ there exist a number $n \in N$, a probability distribution $(a_1, a_2, \ldots, a_n)$ and $x(1), x(2), \ldots, x(n) \in \bar{R}$ such that

$$d(F, \sum_{i=1}^{n} a_i \, \varepsilon_{x(i)}) < \delta .$$

An immediate consequence of the two previous theorems is the following

Corollary. The metric space (Δ, d) is separable.

The results of this note, as well as those of [5], may be regarded as an explicit realization, by elementary methods, of the results in section II.6 of [3], realization which is convenient in the setting of probabilitstic metric spaces ([4]).

Bibliografy

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