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EMBEDDINGS IN ITERATION GROUPS AND SEMIGROUPS WITH NONTRIVIAL UNITS

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0. Introduction.

The usual interpretation of an iteration group (semigroup) as a dynamical (semidynamical) system is given in the following way. Denoting by X the "state space" of a "system" an iteration (semi) group $\{f^t\}$ describes the transition of the states in "time spans" t. Naturally this interpretation leads to the property $f^o(x) = x$ for the neutral element f^o of an iteration (semi) group: after a zero time span the system is still in the same state. This may be the reason why in most work on dynamical systems $f^o(x) = x$ is demanded for all states x. Only recently was it noticed in work of Sklar and Zdun that iteration groups or semigroups without this requirementexhibit interesting properties.

<u>Definition 0.1.</u> Let H be a sub(semi)group of the additive group of reals. We call a family $\{f^t : X \to X;\ t \in H\}$ of mappings an iteration (semi)group if

0.2. $$f^s \circ f^t = f^{s+t} \quad \text{for all } s,t \text{ in } H.$$

An important example is the semigroup of natural iterates. Given a self-mapping fog a set, the natural iterates f^n of f, recursively defined by $f^1 = f$, $f^{n+1} = f \circ f^n$, $n \in \mathbb{N}$, from an iteration semigroup.

As a second example consider a set X consisting of two points $X = \{1,2\}$ and the mapping f of this set into itself with fixed point 2 and $f(1) = 2$. This mapping is idempotent: $f \circ f = f$, and a real iteration group is given by $f^t = f$ for all real t. Clearly the unit element f^o is different from the identity on X.

A real iteration group consisting of continuous functions is given by: $X = [0,2]$, and for real t

$$f^t(x) = \begin{cases} x^{(3^{-t})} & , \quad x \in [0,1] \\ (2-x)^{(3^{1-t})}, & x \in [1,2] \end{cases} .$$

In the following we discuss the question whether the discrete system of natural iterates can be embedded in a time continuous system.

<u>Definition 0.3.</u> Let f be a self-mapping of a set. An iteration (semi)group $\{f^t; t \in H\}$, $1 \in H$ is called an H-embedding of f if

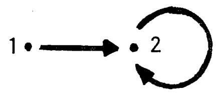
0.4 $$f^1 = f.$$

Due to the property 0.2 the natural iterates f^n of the embedded function coincide with the mappings f^t for $t = n$ of the iteration (semi)group. The semigroup of natural iterates becomes a subsemigroup of the embedding.

A self mapping f of a set X defines an iterative structure on the set. The relation

0.5 $$x \sim_f y \quad \Leftrightarrow \quad \exists\, m,n \in N \quad f^m(x) = f^n(y)$$

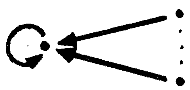
is an equivalence relation; it decomposes X into disjoint classes, the orbits. Orbits can be considered as directed funtional graphs, an edge leading from x to y if and only if $y = f(x)$. The only orbit of the mapping in the second of the above examples is of the form:



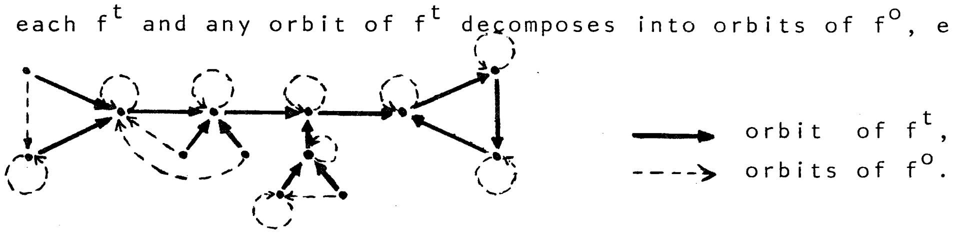
A k-cycle consists of k points mapped into each other cyclically by f, and an orbit may contain at most one cycle. For further information on orbits we refer the reader to chapter 1 in Targonski [7].

1. Nontrivial Units.

By definition, the unit f^o of an iteration (semi)group is idempotent, $f^o \circ f^o = f^o$. For an idempotent mapping the range coincides with the set of fixed points. Thus the orbits consist of a fixed point and its preimages; for example



. Since $f^t = f^t \circ f^o$ for all t and due to the special structure of the orbits of f^o each mapping f^t is constant on orbits of f^o, i.e. any orbit of f^o is contained in a single orbit of each f^t and any orbit of f^t decomposes into orbits of f^o, e.g.:



A further consequence of the (semi)group property 0.2 is $\operatorname{Ran} f^t \subset \operatorname{Ran} f^o$ for all t. For an iteration group we have $f^o = f^t \circ f^{-t}$, i.e. $\operatorname{Ran} f^o \subset \operatorname{Ran} f^t$, and $\operatorname{Ran} f^o = \operatorname{Ran} f^t$ for all t in this case. We summarize these observations in the following lemma.

<u>Lemma 1.1.</u> Let $\{f^t;\ t \in H\}$ be an iteration group or semigroup with a unit. Then

(a) $\operatorname{Ran} f^o = \{x \in X;\ f^o(x) = x\}$,

(b) $\operatorname{Ran} f^t \subset \operatorname{Ran} f^o$ for all $t \in H$, and

$\operatorname{Ran} f^t = \operatorname{Ran} f^o$ for all $t \in H$ if H is a group,

(c) $f^o(x) = x$ for all x in $\bigcup_{t \in H \setminus \{0\}} \operatorname{Ran} f^t$.

With (b) the restrictions f_o^t of f^t to the set $X_o := \operatorname{Ran} f^o$ are mappings into X_o. The (semi)group property 0.2 is valid for these mappings and we have

<u>Lemma 1.2.</u> Let $\{f^t;\ t \in H\}$ be an iteration group or semigroup with a unit. With $X_o := \operatorname{Ran} f^o$ the mappings $f_o^t := f^t|_{X_o}$ form an iteration (semi)group with $f_o^o = \operatorname{Id}_{X_o}$.

If H is a group, the mappings f_o^t are bijections on the set X_o, and f_o^{-t} is the inverse mapping to f_o^t.

Proof. From (a), lemma 1.1 we have $f_o^o = \operatorname{Id}_{X_o}$. If H is a group, then $\operatorname{Id}_{X_o} = f_o^o = f_o^t \circ f_o^{-t} = f_o^{-t} \circ f_o^t$ for all t in H by the group property. Thus the mappings f_o^t are bijective and f_o^{-t} is the inverse of f_o^t.

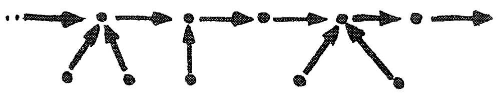
With the last lemma the mappings of iteration groups can be characterized.

<u>Definition 1.3.</u> (Sklar [3]). A self-mapping f of a set is called ultrastable, if the restriction to its range (as a mapping into its range) is bijective.

We can thus reformulate the last part of lemma 1.2 as

<u>Theorem 1.4.</u> The mappings of an iteration group are ultrastable.

The orbits of ultrastable mappings are of the following form. On its range these mappings are bijective, i.e. the restrictions of the orbits to the range are either cycles or chains, ordered like $\mathbb{Z}$. The elements outside the range are mapped into these cycles or chains in one step. Thus the orbits look like: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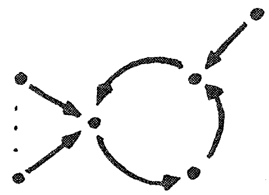


We shall now discuss the problem what mappings can be added as a neutral element to an iteration semigroup without a unit (there exists always such a mapping, the identity). An equivalence relation on X is defined by $a \sim b \Leftrightarrow f^t(a) = f^t(b)$ for all t in H, the equivalence classes are denoted by [a]. Suppose f^o is a unit. Since f^o has to satisfy $f^t = f^t \circ f^o$ for all t, it maps each equivalence class into itself. The mapping f^o is idempotent, thus each class contains at least one fixed point of f^o, and points that are not fixed points are mapped by f^o on fixed points in the same class. Thus all neutral elements can be constructed in the following way.

<u>Lemma 1.5.</u> Let $\{f^t;\ t \in H\}$ be an iteration semigroup without a unit element. An idempotent mapping e, that maps each equivalence class into itself, such that the set of fixed points of e contains $\bigcup_{t \in H} \operatorname{Ran} f^t$, can be added as a unit to the iteration semigroup.

Under two conditions the identity mapping is the only possible neutral element.

<u>Lemma 1.6.</u> (cf. Korollar (1.2.3) in Graw [2]). Let $\{f^t;\ t \in H\}$ be an iteration semigroup without a unit element. If for one t in H the mapping f^t is either injective or surjective, then the only possible unit element is the identity mapping.

Proof. If one of the mappings f^t is surjective, then $\bigcup_{t \in H} \operatorname{Ran} f^t = X$, thus with lemma 1.1c. each neutral element is the identity mapping. Let f^o be a neutral element and let the mapping f^t be injective for a t in H. Suppose $y = f^o(x)$, then $f^t(x) = f^t \circ f^o(x) = f^t(y)$, thus x=y, since f^t is injective. Therefore $f^o = Id$.

2. Embeddings.

This paragraph consists of three parts. In the first we introduce iterative roots, a notion that is unseparably connected with the problem of embedding. We then prove a theorem of Tabor on generating a rational embedding by iterative roots. In the second part we consider the role specially chosen neutral elements for the semigroup of natural iterates play for the problem of embedding. In the last part we give a proof for a condition of Sklar for embeddability in a real iteration group.

Given a mapping f of a set X into itself and a natural number $n \geqslant 2$, we call a self-mapping g of X an n-th iterative root of f and denote it by $f^{1/n} := g$, if $g^n = f$. The problem of finding an iterative root was only recently solved by Zimmermann [9]. Some of these results can be found in Targonski [7]. Now suppose $\{f^t\}$ is a rational or real embedding of the mapping $f = f^1$. Then for all natural numbers $n \geqslant 2$ the mapping f^t with $t = 1/n$ is an n-th iterative root of f. This can be verified by using the (semi)group property (0.2) n-1 times. We thus have proved.

Theorem 2.1. For the embeddability of a mapping f it is necessary that there exist iterative roots of f of all orders.

For our further discussion we need conditions for the existence of iterative roots of bijections. The orbits of bijections are either cycles, or chains ordered like $\mathbb{Z}$. Denoting by L_o the (possibly infinite) number of chains, and by L_k the (possibly infinite) number of k-cycles we have.

Theorem 2.2. of Łojasiewicz. A bijective self-mapping of a set has an n-th iterative root if and only if for every $k \in \mathbb{N}_o$, $L_k = \infty$ or L_k is divisible by d_k. Here $d_o = n$, and for $k > 0$, $d_k = n/n_k$, n_k being the greatest divisor of n relative prime to k.

The theorem does not restrict the number L_1 of fixed points, since $d_1 = 1$ and L_1 is always divisible by 1.

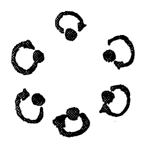
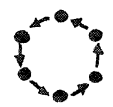
In view of 2.1 we need conditions for the existence of roots of all orders. The preceding theorem yields

Theorem 2.3. Let f be a bijective self-mapping of a set. For the existence of iterative roots of f of all orders it is necessary and sufficient that $L_k = 0$ or $L_k = \infty$ for all $k \in N_o \setminus \{1\}$.

Proof. Necessity. Let $k \neq 1$ be given and assume that $0<L_k<\infty$. If $k = 0$, we shall show that f has no root of order $n = 2 \cdot L_o$. In the theorem of Łojasiewicz $d_o = n = 2 \cdot L_o$ and L_o is not divisible by $2 \cdot L_o$ since $L_o>0$. Thus there exists no $2 \cdot L_o$-th root.

If $k \geqslant 2$ the mapping f has no root of order $n = k^{L_k}$. In fact $(n,k) = k$, $n_k = 1$, therefore $d_k = n = k^{L_k} > L_k$, thus d_k does not divide L_k, and there exists no iterative root of order k^{L_k}. The sufficiency follows directly from the preceeding theorem.

We have seen in 2.1 that the existence of iterative roots of all orders is a necessary condition for the embeddability of a mapping. Is it also sufficient? The problem is, how a given sequence of iterative roots generates a rational embedding. If for example the mappings f, $f^{1/2}$, and $f^{1/6}$ are given like

f:  , $f^{1/2}$:  , $f^{1/6}$: 

then the third natural iterate of $f^{1/6}$ (being an iterative squareroot) should be equal to the given square root $f^{1/2}$. But clearly these two mappings are different. Thus to generate a rational embedding we have to choose a set of iterative roots that is independent in the sense that no contradictions as above occur. One way to solve this problem is not to use a generating set consisting of roots of all orders but to choose a generating sequence such that an element of this sequence is an iterative root of its predecessor.

Theorem 2.4 of Tabor [6]. Let f be a self-mapping of a set, $\{r_i\}$ a sequence of natural numbers and

$f^{1/r_1}, (f^{1/r_1})^{1/r_2}, \ldots, (\ldots(f^{1/r_1})\ldots)^{1/r_n}$ a sequence of successive iterative roots. If for any natural number n there is an $i \in N$ with $n | r_1 \cdot \ldots \cdot r_i$, then the sequence of roots generates an embedding of f in a rational iteration semigroup.

Proof. Abbreviation: $(\ldots(f^{1/r_1})\ldots)^{1/r_n} =: f^{1/r_1\ldots r_n}$. Under the premisses of the theorem there are $m, i \in N$ with $m \cdot n = r_1 \cdot \ldots \cdot r_i$ for every positive rational number $r = p/n, p, n \in N$, thus $p/n = mp/r_1 \cdot \ldots \cdot r_i$. We define a mapping $f^r : X \to X$ by $f^r := (f^{1/r_1\ldots r_i})^{m\cdot p}$ (m·p-th natural iterate of $f^{1/r_1\ldots r_i}$). This definition does neither depend on the special representation $r = p/n$ of the rational number nor the special choice of the number i. To check the "time-one" condition 0.4 we can choose $i=1$, $m = r_1$, $r = 1$ and we have by definition $f^1 = f^r = (f^{1/r_1})^{r_1} = f$. For two numbers $r_1, r_2 \in Q^+$, $r_j = p_j/n_j$ with natural numbers p_j, n_j for $j = 1,2$ there exist natural numbers i,m such that $n_1 n_2 | r_1 \cdot \ldots \cdot r_i$, $n_1 n_2 m = r_1 \cdot \ldots \cdot r_i$. Setting $m_1 = mn_2$, $m_2 = mn_1$ we have by definition

$$f^{r_1} \circ f^{r_2} = (f^{1/r_1\ldots r_i})^{m_1 p_1} \circ (f^{1/r_1\ldots r_i})^{m_2 p_2}$$

$$= (f^{1/r_1\ldots r_i})^{m_1 p_1 + m_2 p_2} \quad \text{(composition of natural iterates of } f^{1/r_1\ldots r_i})$$

$$= (f^{1/r_1\ldots r_i})^{m(n_2 p_1 + n_1 p_2)}$$

$$= f^{r_1 + r_2}, \text{ since } r_1 + r_2 = \frac{n_2 p_1 + n_1 p_2}{n_1 n_2}$$

Therefore $\{f^r;\ r \in Q^+\}$ is a Q^+-embedding of f. Since every member f^r is a finite composition of members of the original sequence, the embedding is generated by this sequence.

Remark. As we have seen in the proof, the condition in Tabor's theorem guarantees that the sequence $1/r_1$, $1/r_1r_2$ generates the additive semigroup of positive rational numbers.

The theorem of Tabor does not answer the question whether a mapping having roots of all orders is Q^+-embeddable. This is still an open question. For embeddings in rational groups the problem is solved.

Theorem 2.5. For the embeddability of a bijection in a rational iteration group it is necessary and sufficient that $L_k = 0$ or $L_k = \infty$ for all $k \neq 1$.

Proof. The necessity follows directly from theorems 2.1 and 2.2. Let $L_k = 0$ or $L_k = \infty$ for all $k \neq 1$. The sequence $\{r_i\}$ with $r_i = i+1$ fulfills the conditions of the theorem of Tabor. We know from the theorem of Łojasiewicz that f possesses an r_1-th iterative root f^{1/r_1}. It is now necessary to cite a result from the work of Zimmermann [9].

2.6. An n-th iterative root g of a bijection is bijective. The orbits of g are unions of a finite number of orbits of f. A cycle of g is a union of cycles (all with the same order) of f (the order of the f-cycles does only depend on n and the order of the g-cycle), and a chain of g is the union of chains of f.

Thus we have the following orbit structure of f^{1/r_1}. If $L_o=0$, then there are no chains of f^{1/r_1}, if $L_o = \infty$ then there exists infinitely many chains of f^{1/r_1}, since each of them is a union of only finitely many chains of f. If $L_k = \infty$ for some $k \geqslant 2$, these cycles of f give rise to infinitely many cycles of f^{1/r_1} with a certain order k'. Thus the mapping f^{1/r_1} fulfills the assumptions of theorem 2.3 and has therefore an iterative root of order r_2: $(f^{1/r_1})^{1/r_2}$. The same argument applies to this function and we obtain a sequence of consecutive iterative roots that genera-

one acyclic orbit of f and since the cycles of f_o^1 are the cycles of f we get the necessary condition of theorem 2.9.

Let now f be ultrastable and fulfill the conditions on the number of orbits. We choose a neutral element e according to Lemma 2.8 for the semigroup of natural iterates with Ran e = Ran f. The bijective mapping $f_o := f|_{\text{Ran } f}$ fulfills the criterion on the numbers of orbits in Theorem 2.5, since the chains of f_o are the restrictions of the acyclic orbits of f to Ran e, and the cycles of f_o and f coincide. Thus f_o is embeddable on Ran e, and with Lemma 2.7 the mapping f is embeddable in a rational iteration group.

We now use a method of Sklar [5] for reindexing a given rational iteration group as a real iteration group. Let B be a Hamel basis for the real numbers containing 1. For a real number t let h(t) be the coefficient of the basis element 1 in the expansion of t with respect to the given Hamel basis, i.e. if $t = r_1 \cdot 1 + r_2 \cdot b_2 + \ldots + r_n \cdot b_n$ for some $n \in N$, $b_2, \ldots, b_n \in B$ and $r_1, \ldots, r_n \in Q$, then $h(t) = r_1$. Since the expansion is unique, we have h(r) = r for rational r and the mapping h is a group homomorphism from (R,+) onto (Q,+). Let $\{f^r;\ r \in Q\}$ be a rational iteration group, then the mappings $f^t := f^{h(t)}$ for real t form a real iteration group that contains the given rational group. We have thus proven

Theorem 2.10 (Sklar [3]). For a self mapping of a set to be embeddable in a real iteration group it is necessary and sufficient that the mapping be ultrastable, the number of its acyclic orbits is either 0 or ∞, and the number of its k-cycles, $k \geq 2$, is either 0 or ∞.

It is not (in general) possible to extend a given Q^+-iteration semigroup to an R^+-semigroup in an analogous way, since there exists no Hamel basis for the positive real numbers (Aczél, Erdős [1]).

3. Embedding in One Dimensional Zdun Semi-Flows.

In this section we turn our attention to iteration groups and semi-groups that satisfy certain continuity conditions. Extensively studied (Zdun [8], the main results can be found in Targonski [7], chap. 3) have been "continuous iteration (semi)groups", CIS or CIG for short, consisting of continuous functions f^t on an interval such that the mappings $t \to f^t(x)$ are continuous for all x in the interval (see the third example in the introduction for an example of a CIG). We shall discuss iteration semigroups on intervals demanding continuity only in the time parameter. On certain subintervals the continuity in t will imply the continuity in x, i.e. the restriction of the semigroup to such a subinterval is a CIS. This is a feature due to the special structure of $\mathbb{R}$ and an analogous result cannot be expected in higher dimensions. The following definition is due to Sklar [4].

<u>Definition 3.1.</u> Let I be a closed interval. An iteration semigroup $\{f^t;\ t \geq 0\}$ is called a one dimensional Zdun semi-flow if for x in Ran f^0 the function h_x: $[0,\infty) \to I$ defined by $h_x(t) := f^t(x)$ is continuous.

Example. $I = [-1,1]$,

$$f^t(x) = \begin{cases} 2^{-t}x & , x \neq 0, \\ 2^{-1-t}x & , x = 0. \end{cases} \qquad t \geq 0.$$

In the following theorems we are going to characterize the functions in a one dimensional Zdun semi-flow.

<u>Theorem 3.2.</u> (Sklar [4]).

(a) Every function f^t is nondecreasing on the set Ran f^0,

(b) let x be in Ran f^0. If there is an $s>0$ such that $f^s(x) = x$, then $f^t(x) = x$ for all $t \geq 0$, and the function h_x is constant,

(c) for any x in Ran f^o, either h_x is strictly monotonic on $[0,\infty)$, or there is an $s \geqslant 0$ such that h_x is strictly monotonic on $[0,s]$ and $h_x(t) = h_x(s) = f^t(h_x(s))$ for all $t \geqslant s$.

Remark. The second part states that a fixed point x of one of the functions f^s, $s>0$ is already a stationary point of the flow (i.e. $f^t(x) = x$ for all t). This rules out the possibility of closed orbits for the flow. This is intuitivaly clear, a closed orbit of a flow must be homeomorphic to the circle, but there exist no subsets of R homoemorphic to a circle.

Proof. (a) Let x,y be in Ran f^o with $x<y$, then with 1.1 $f^o(x) = x$ and $f^o(y) = y$. Assume that for a $t>0$, $f^t(x)>f^t(y)$. The mappings h_x and h_y are continuous with $h_x(0)=x<y = h_y(0)$ and $h_x(t)>h_y(t)$. Thus there exists an $s\in(0,t)$ with $h_x(s) = h_y(s)$. It follows that $f^t(x) = f^{t-s}(f^s(x)) = f^{t-s}(f^s(y)) = f^t(y)$ in contradiction to the assumption. Thus the mappings f^t are nondecreasing on Ran f^o.

(c) Let x be in Ran f^o and suppose that h_x is not strictly monotonic. Then there are $s,t\in[0,\infty)$, $s<t$ with $h_x(s) = h_x(t) =: y$. Then with the semigroup property 0.2 $T := t-s$ is a period of the function h_x restricted to $[s,\infty)$. It can be seen that the positive periods of h_x restricted to $[s,\infty)$ form a semigroup that is either equal to R_o^+, has a smallest element, or is dense in R_o^+. By the continuity of h_x the third possibility is ruled out. Now assume that h_x is not constant on $[s,\infty)$. Then there exists a smallest period $\tau>0$ of h_x restricted to $[s,\infty)$. Then h_x is injective on $[s,s+\tau)$, since otherwise τ would not be minimal. Thus h_x is strictly monotonic on $[s,s+\tau)$. But h_x is continuous on $[s,\infty)$ with $h_x(s) = h_x(s+\tau)$, therefore h_x is constant on $[s,\infty)$.

Let $s_o \geqslant 0$ be the smallest number such that h_x is constant on $[s_o,\infty)$. If $s_o = 0$, then (c) is proved. Let $s_o>0$ and assume $h_x(u) = h_x(v)$ for $u,v\in[0,s_o]$, $u<v$, then $h_x \equiv y$ on $[u,\infty)$ with the same argument as above - contradiction to the minimality of s_o. Thus h_x is strictly monotonic on $[0,s_o]$.

(b) Let x be in Ran f^o. Then $f^o(x) = x$ with 1.1, and $h_x(0)=h_x(s)=x$. With (c) the function h_x is constant on $[0,\infty)$.

Definition 3.3. Let x be in Ran f^o. The set $T(x) := \{f^t(x);\ t \geqslant 0\}$ is called the trajectory of x, and the set $T_c(x) := T(x) \cup \{z \in \text{Ran } f^o;\ x \in T(z)\}$ the complete trajectory of x.

With the properties in Theorem 3.2 it can be seen that two different complete trajectories are either disjoint or have exactly one point in common, and this point is a fixed point. In the sense of the interpretation in the introduction, the trajectory of a state contains the "future", and the complete trajectory, in addition, the "past" of this state.

Lemma 3.4.

(a) Let x be in Ran f^o. Then $T(x)$ and $T_c(x)$ are invariant under the mappings f^t.

(b) Let x be in Ran f^o with $f^1(x) \neq x$. Then $T(x)$ and $T_c(x)$ are intervals.

(c) Let x be in Ran f^o with $f^1(x) \neq x$. Then the functions f^t are continuous on $T_c(x)$.

Proof. (a) Let $t \geqslant 0$ and $y \in T(x)$. Then there is an $s \geqslant 0$ with $y = f^s(x)$ and we have $f^t(y) = f^t(f^s(x)) = f^{s+t}(x) \in T(x)$, thus $T(x)$ is invariant under f^t. Now let y be in $\{z \in \text{Ran } f^o;\ x \in T(z)\}$, then $x = f^s(y)$ for some $s \geqslant 0$, and we have $x = f^s(y) = f^{s-t}(f^t(y))$, i.e. $f^t(y) \in \{z \in \text{Ran } f^o;\ x \in T(z)\}$, if $t<s$, and $f^t(y) = f^{t-s}(f^s(y)) = f^{t-s}(x) \in T(x)$ if $t \geqslant s$. Thus $T_c(x)$ is invariant under f^t.

(b) The trajectory $T(x)$ is an interval, since it is the image of the interval $[0,\infty)$ under the continuous mapping h_x.

Now suppose $f^1(x)>x$. Then h_x is increasing (cf. (c), Th. 3.2).

(1) Let h_x be strictly increasing. Then $T(x) = [x,y)$ with $y = \lim_{t\to\infty} f^t(x)$. With the semigroup property and theorem 3.2 it can be shown, that all h_z, $z \in T_c(x)$ are strictly increasing and that

$T(z) = [z,y)$. Since $T_c(x) = T(x) \cup \bigcup_{x \in T(z)} T(z)$ the complete trajectory is an interval that is open on the right side with end point y.

(2) Let h_x be strictly increasing on $[0,s]$ for some $s \geq 0$ and constant on $[s,\infty)$. Then $T(x) = [x,y]$ with $y = h_x(s)$. It can be shown that h_z, for $z \in T_c(x)$ is strictly increasing on $[0,s_z]$ for some $s_z \geq 0$, constant on $[s_z,\infty)$, and $T(z) = [z,y]$. Thus the complete trajectory is an interval that is closed on the right with end point y.

In the case $f^1(x)<x$, i.e. h_x is decreasing, we get analogous results by interchanging "left" and "right".

(c) Suppose $f^1(x)>x$ (the argument for $f^1(x)<x$ is analogous) and assume for simplicity that the interval $T_c(x)$ is closed on the left side $T_c(x) = [z,y)$ resp. $[z,y]$.

(1) Let h_z be strictly increasing, i.e. $T(z) = [z,y)$ with (b). Then there exists the inverse mapping $h_z^{-1}: T(z) \to [0,\infty)$ and this mapping is continuous. Then we have for $y \in T_c(x)$ and $t \geq 0$

$$f^t(y) = f^t(h_z(h_z^{-1}(y))) = f^t(f^{h_z^{-1}(y)}(z)) = h_z(t+h_z^{-1}(y)).$$

Thus f^t is continuous as a composition of continuous mappings.

(2) Let h_z be strictly increasing on $[0,s]$ and constant on $[s,\infty)$ for some $s \geq 0$. Then there exists a continuous inverse $h_z^{-1}:T(z) \to [0,s]$ of h_z restricted to $[0,s]$. With the same argument as in (1) the functions f^t are continuous on $T(z) = T_c(x)$.

We can now characterize the functions of a Zdun semi-flow. The set Ran f^o is a union of complete trajectories and fixed points. The form of the mappings f^t on each trajectory is described in

<u>Theorem 3.5.</u> Let x be in Ran f^o with $f^1(x) \neq x$. Then the functions f^t, $t \geq 0$ are of one of the following forms on the complete trajectory (all functions f^t are of the same form).

p1(a) $T_c(x) = (y,b]$ or (y,b), $\lim_{z\to y+} f^t(z) = y$, $\lim_{z\to b-} f^t(z)<b$,

f^t strictly increasing on $T_c(x)$,

(b) $T_c(x) = (a,y)$ or $[a,y)$, $\lim_{z\to y-} f^t(z) = y$, $\lim_{z\to a+} f^t(z)>a$,

f^t strictly increasing on $T_c(x)$,

p2(a) $T_c(x) = [y,b)$ or $[y,b]$, $f^t(y) = y$, $\lim_{z\to b-} f^t(z)<b$,

there exists a $c_t \in (y,b]$ with $f^t|_{[y,c_t]} \equiv y$ and f^t is

strictly increasing on $[c_t,b)$ resp. $[c_t,b]$.

(b) $T_c(x) = (a,y]$ or $[a,y]$, $f^t(y) = y$, $\lim_{z\to a+} f^t(z)>a$,

there exists a $c_t \in [a,y)$ with $f^t|_{[c_t,y]} \equiv y$ and f^t is

strictly increasing on $(a,c_t]$ resp. $[a,c_t]$.

p3(a) $T_c(x) = (a,b)$, f^t strictly increasing on (a,b),

$\lim_{z\to a+} f^t(z) = a$, $\lim_{z\to b-} f^t(z) = b$.

p4(a) $T_c(x) = [y,b)$, $\lim_{z\to b-} f^t(z) = b$, $f^t(y) = y$,

there exists a $c_t \in (y,b)$ with $f^t|_{[y,c_t]} \equiv y$ and f^t is strictly increasing on $[c_t,b)$,

(b) $T_c(x) = (a,y]$, $\lim_{z\to a+} f^t(z) = a$, $f^t(y) = y$,

there exists a $c_t \in (a,y)$ with $f^t|_{[c_t,y]} \equiv y$ and f^t is strictly increasing on $(a,c_t]$.

Proof. We only prove two exemplary cases. Assume $f^1(x)>x$, i.

e. h_x is increasing, and let $T_c(x)$ be closed on the left side, $T_c(x) = [a,y)$ resp. $[a,y]$.

(1) Let h_x be strictly increasing, $T_c(x) = [a,y) = T(a)$. As we have seen in the proof of (3.4c), $f^t(y) = h_a(t+h_a^{-1}(y))$, thus f^t is strictly increasing as a composition of two strictly increasing mappings. Since f^t is strictly increasing the limit

$\lim_{z\to y-} f^t(z) =: w$ exists.

Thus we have $w = \lim_{s\to\infty} f^t(f^s(a))$, since $\lim_{s\to\infty} f^s(a) = y-$ and we have $w = \lim_{s\to\infty} f^{t+s}(a) = y$. This proves that f^t is of the form P1(b).

(2) Let h_a be strictly monotonic on $[0,s]$ and constant on $[s,\infty)$ for some $s \geqslant 0$. Let $h_a^{-1}: T_c(x) \to [0,s]$ be the inverse mapping of h_a restricted to $[0,s]$. Then $f^t(z) = h_a(t+h_a^{-1}(z))$ for $t \geqslant 0$ and $z \in T_c(x)$. If $t+h_a^{-1}(z) \geqslant s$ for all $z \in T_c(x)$ then f^t is constant on the complete trajectory, i.e. of form P2(b). Otherwise there is an $u \in T_c(x)$ with $t+h_a^{-1}(z) \leqslant s$ for $z \in [a,u]$, and $t+h_a^{-1}(z) \geqslant s$ for $z \in [u,y]$. Thus f^t is strictly increasing on $[a,u]$ and constant on $[u,y]$. Since $f^t(y) = f^t(f^s(a)) = f^{s+t}(a) = y$, y is a fixed point of f^t and f^t is of the form P2(b).

We now turn to the problem of embedding a function in a one dimensional Adun semi-flow.

<u>Theorem 3.6.</u> Let f be a self mapping of a closed interval I, I_o a subset of I such that $f(I) = f(I_o) \subset I_o$. If there is a decomposition of I_o into intervals having at most a fixed point of f in common and fixed points such that f is of one of the forms P1-P4 of Th. 3.5 on each of these intervals, then f is embeddable in a one dimensional Zdun semi flow.

Proof. With the condition on the set I_o and Lemma 1.5 there exists a unit f^o for the semigroup of natural iterates of f with Ran $f^o = I_o$. We are going to construct an embedding of f separately in each of the intervals. We only demonstrate the construction in the case P1(b) and refer the reader to Zdun [8] for the other cases. Let f be of the form P1(b) on the interval J and assume J is closed on the left side, $J = [a,y)$. We are going to construct an invertible solution $A:J \to [0,\infty)$ of the Abel equation

$$A(f(x)) = A(x) + 1 \quad \text{for} \quad x \in J. \tag{3.7}$$

Let $A_o: [a,f(a)] \to [0,\infty)$ be an arbitrary continuous, strictly increasing function with $A_o(f(a)) = A_o(a) + 1$. Since $J = [a,y) = \bigcup_{n \in N_o} [f^n(x), f^{n+1}(x))$ is a disjoint union, one can define a continuous function $A: J \to [0,\infty)$ by $A(x) := A_o(x)+n$ if $x \in [f^n(x), f^{n+1}(x))$ for some $n \in N_o$. This mapping satisfies (3.7), is strictly increasing and surjective.

Setting $f^t(x) := A^{-1}(t+A(x))$ for $x \in J$ and $t \geqslant 0$, we have defined an embedding of f restricted to J. Since A and A^{-1} are continuous, the embedding is a Zdun semi-flow.

In this way we have constructed an embedding of f on the set I_o (if there are any fixed points outside the intervals we put $f^t(x)=x$ for all $t \geqslant 0$). With lemma 2.7. we can use the specially chosen f^o to construct an embedding on the whole interval I.

If we extend the notion of Zdun semi flow to Zdun flow in the obvious way, this leaves only P3 as the possible form of the mappings f^t on each complete trajectory, and if permit only the form P3 in theorem 3.6, then the function f is embeddable in a Zdun flow.

We shall now describe the orbit structure of functions of the forms P1 - P4.

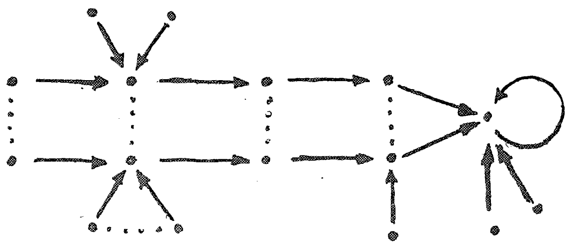
P1 There exist uncountably many chains ordered like N.

P2 The whole interval is one single orbit, since each point reaches the fixed point in finitely many steps. The orbit consists of a fixed point with uncountably many, unramified branches or either the same finite length, or uncountably many of length n, and uncountably many of length n+1 for some $n \in \mathbb{N}$.

P3 There exist uncountably many chains ordered like $\mathbb{Z}$.

P4 The whole interval is one single orbit. It consists of a fixed point with infinitely many, infinitely long, unramified branches.

The orbits just described are restrictions of orbits of f^t to Ran f^o. As we have seen in section 1, points in $I \setminus$Ran f^o are mapped into Ran f^o in one step by f^t. In the cases P3 and P4 each point of the orbit restricted to the complete trajectory may have arbitrary many preimages in $I \setminus$Ran f^o. In the cases P1 and P2, only points having a preimage in the complete trajectory under f^t may have arbitrary many preimages in $I \setminus$Ran f^o (cf. lemma 1.5). For example in case P2:



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