NOTAS BREVES

STOCHASTICA Vol. VII, Nº 2, 1983

# A NOTE ON THE CONSTRUCTION OF MEASURES TAKING THEIR VALUES IN A BANACH SPACE WITH BASIS

Maria Congost Iglesias

*ABSTRACT*

*If E is a Banach space with a basis $\{e_n\}_{n\in N}$, a vector measure $m : \mathfrak{a} \to E$ determines a sequence $\{m_n\}_{n\in N}$ of scalar measures on $\mathfrak{a}$ named its components. We obtain necessary and sufigicent conditions to ensure when given a sequence of scalar measures it is possible to construct a vector valued measure whose components were those given. Furthermore we study some relations between the variation of the measure m and the variation of its components.*

A sequence of elements in a set X will be indicated by $\{x_n\}_{n\in N}$ or $\{x_n\}$ simply. In a normed space it will be used $\Sigma x_n$ instead of $\sum_{n\in N} x_n$ to indicate a series. For the results from: measure theory used in this paper, see (2).

Let $(E,\|\|)$ be a real of complex Banach space. The sequence $\{e_n\}$ is said to be a basis for E if for every $x\in E$ there is a unique sequence of scalar numbers $\{x_n\}$ such that $x=\Sigma x_n e_n$, this series beeing convergent with respect to the norm topology in E. Let $\pi_n$ be the n-projection on E defined by $\pi_n(x)=x_n$. If $m: \mathfrak{a} \to E$ is a vector measure on a $\sigma$-field $\mathcal{a}$ of subsets of X, we consider the set $\{m_n\}$ of scalar measures on E defined by $m_n=\pi_n \circ m(n\in N)$. Then $m(A) = \Sigma m_n(A)e_n (A\in \mathcal{a})$.

The total variation of m is the real measure V(m) on $\mathcal{a}$ defined by

$$v(m)(A)=\sup\{\sum_{i\in I}\|m(A_i)\|\},$$

where the supremum is taken over all finite partitions $\{A_i\}_{i\in I}$ of A on $\mathcal{a}$ (that is to say: I is an arbitrari finite index set and $A_i\in \mathcal{a}$, $A_i \cap A_j = \emptyset$ if $i\neq j$ and $\bigcup_{i\in I} A_i = A$, whenever $i,j\in I$). In the same way $V(m_n)$ will indicate the total variation of $m_n$. All the measures $m, m_n$ and $V(m_n)$, $n\in N$, are bounded (2) but V(m) may be not finite (2,3). Let us consider the real numbers $M=\sup\{\|m(A)\|; A\in \mathcal{a}\}$ and $M_n = \sup\{|m_n(A)|;\ A\in \mathcal{a}\}$, $n\in N$.

<u>Proposition 1.</u> a) $V(m)(A) \leqslant \Sigma V(m_n)(A)\|e_n\|$ $(A\in \mathcal{a})$;

b) If $\inf\{\|e_n\|;\ n\in N\} \neq 0$, then there is a constant $K>0$ such that, for every $n\in N$.

(i) $M_n \leqslant KM$,

(ii) $V(m_n) \leqslant KV(m)$.

Proof. For every Banach space with basis there is a constant $H>0$ such that

$$\|x\| \leqslant \Sigma|x_n|\|e_n\| \quad \text{and} \quad |x_n| \leqslant \frac{H}{\|e_n\|}\|x\|$$

for each $x\in E$. Hence,

$$M_n \leqslant \frac{H}{\|e_n\|} M \quad \text{and} \quad V(m_n) \leqslant \frac{H}{\|e_n\|} V(m);$$

if $a=\inf\{\|e_n\|;\ n\in N\}\neq 0$, for $K=\frac{H}{a}$, we obtain (i) and (ii). Now, let $\{A_i\}_{i\in I}$ be a finite partition of a set A on $\mathcal{a}$. Since

$$\sum_{i\in I}\|m(A_i)\| \leqslant \sum_{i\in I}\sum_{n\in N}|m_n(A_i)|\|e_n\| \leqslant \sum_{n\in N}\|e_n\|\sum_{i\in I}|m_n(A_i)| \leqslant \sum_{n\in N}\|e_n\|V(m_n)(A),$$

we have $V(m)(A) \leqslant \Sigma V(m_n)(A)\|e_n\|$.

A vector measure $m:\mathcal{a} \to E$ determines a sequence $\{m_n\}$ of scalar measures on $\mathcal{a}$. We may ask if a sequence of scalar measures on $\mathcal{a}$ determines a vector measure or not (the space E and the basis $\{e_n\}$ are supposed fixed). By the uniqueness of the expression $x = \Sigma x_n e_n$, it is necessary that $m(A) = \Sigma m_n(A)e_n$, that is to say, the series $\Sigma m_n(A)e_n$ must be convergent, whenever $A\in\mathcal{a}$. This is also sufficient:

<u>Theorem 1.</u> Let E be a Banach space with basis $\{e_n\}$. If $\{m_n\}$ is a sequence of scalar measures on a $\sigma$-field $\mathcal{a}$ such that $\Sigma m_n(A)e_n$ is convergent in E for every $A\in\mathcal{a}$. Then the set function $m:\mathcal{a} \to E$ defined by $m(A) = \Sigma m_n(A)e_n$ is a vector measure on $\mathcal{a}$ such that $m_n=\pi_n\circ m$ for every $n\in N$.

Proof. For every $n\in N$, we can consider the set function $S_n:\mathcal{a} \to E$ defined by $S_n(A) = \sum_{i=1}^{n} m_i(A)e_i$. Thus $m(A)=\lim S_n(A)$ and $(x'\circ m)(A)=\lim(x'\circ S_n)(A)$ for every $x'\in E'$, so that, by the Vitali-Hahn-Sacks theorems $x'\circ m$ is a scalar measure for every $x'\in E'$. Hence, by the Pettis theorem, m is a vector measure. Because of the uniqueness of the expression of an element in a Banach space with basis, we conclude that $m_n=\pi_n\circ m$, $n\in N$.

The next theorem gives us a sufficient condition in order to obtain a vector measure from a sequence of scalar measures:

<u>Theorem 2.</u> Let E be a Banach space with basis $\{e_n\}$. If $\{m_n\}$ is a sequence of scalar measures on a $\sigma$-field $\mathcal{a}$ such that $\Sigma M_n\|e_n\|$ is convergent, then the set function $m:\mathcal{a} \to E$ defined by $m(A)=\Sigma m_n(A)e_n$ is a vector measure of bounded variation; furthermore the series defining m is absolutely and uniformly convergent on $\mathcal{a}$.

Proof. The series $\Sigma m_n(A)e_n$ is clearly absolutely and uniformly convergent so that, by the previous theorem, defines a vector measure m. Since $V(m_n)\leq 4\, M_n$ (see (2)), we have $V(m)\leq \Sigma V(m_n)\|e_n\| \leq 4\, \Sigma M_n\|e_n\|<+\infty$, and m is of bounded variation.

This note ends with two examples showing that the inequalities obtained in Prop. 1 cannot be improved. Morever, the first one shows that $\Sigma M_n\|e_n\|<+\infty$ is not necessary to obtain a vector measure of bounded variation.

Example 1. Let $E=c_o$ with basis $\{e_n\}$, where $e_n=(0,\overset{n}{\ldots},0,1,0,\ldots)$. Let $\mathcal{a}$ be a $\sigma$-field of subsets of a non empty set X and $m_o:\mathcal{a}\to R_+$ a finite measure such that $m_o\neq 0$. Since $m_o\geq 0$, we have $V(m_o)=m_o$. Let $a=\Sigma a_n e_n$ be an element in $c_o$ with $a_n\geq 0$ for all $n\in \mathbb{N}$, and $\Sigma a_n=+\infty$. Let us consider the real measures on $\mathcal{a}$, $m_n=a_n m_o$. Then the set function $m:\mathcal{a}\to c_o$ defined by $m(A)=\Sigma m_n(A)e_n$ is a vector measure on $\mathcal{a}$, because of $\Sigma\, m_n(A)e_n=m_o(A)\Sigma a_n e_n$ is a convergent series in $c_o$. Furthermore, if $\{A_i\}_{i\in I}$ is a finite partition of A, then

$$\sum_{i\in I}\|m(A_i)\|=\sum_{i\in I}\sup_n|m_n(A_i)|=\sum_{i\in I}\sup_n|a_n||m_o(A_i)|\leq$$

$$\sup_n|a_n|.V(m_o)(A)=\sup_n|a_n|.|m_o(A)|=\|m(A)\|,$$

so that $V(m)(A)\leq\|m(A)\|$ and m is of bounded variation. However because of $M_n=\sup\{|m_n(A)|;A\in\mathcal{a}\}=|a_n|.\sup\{|m_o(A)|,\ A\in\mathcal{a}\}=a_nM_o$ and $\Sigma a_nM_o=+\infty$ $(M_o\neq 0)$, the series $\Sigma M_n\|e_n\|$ is not convergent.

With respect to the inequalities in Prop. 1 we observe that, for the measure in the example, we have $V(m)(A)\neq\Sigma V(m_n)(A)\|e_n\|$ in a) while in b(ii) we have just the equality, with K=1, because of $\sup V(m_n)(A)=\sup|m_n(A)|=\|m(A)\|$ and $\|m(A)\|\leq V(m_n)(A)$. But, generally, we can only obtain the inequality in b(ii) as the following example shows.

Example 2. Let $E=\ell_1$ with basis $\{e_n\}$, where $e_n=(0,\overset{n}{\ldots},0,1,0,\ldots)$. Let $\{p_n\}$ be a sequence of elements in a set $X\neq\emptyset$. Let $\mathcal{a}=P(x)$ and

$\{a_n\}$ a sequence of real numbers, two of them nonzero at least, such that $\Sigma|a_n|<+\infty$. For every $n\in\mathbb{N}$. Let $m_n$ be the measure weighted $a_n$ at $p_n$:

$$m_n(A) = \begin{cases} a_n, & \text{if } p_n\in A, \\ 0, & \text{if } p_n\notin A. \end{cases}$$

Since $\Sigma M_n\|e_n\|=\Sigma|a_n|<+\infty$, we have that $m(A)=\Sigma m_n(A)e_n$ is a vector measure $m:\mathcal{a}\to\ell_1$ of bounded variation. From $V(m_n)(A)=|m_n(A)|$ and Prop. 1, a), we see that $V(m)(A)\leqslant\Sigma V(m_n)(A)=\Sigma|m_n(A)|=\|m(A)\|$. Thus $\|m(A)\|=V(m)(A)=\Sigma V(m_n)(A)$. To see that there is no constant $K>0$ such that $\sup V(m_n)=K\ V(m)$ we can observe that, since $V(m_n)(p_i)=|a_i|$ and $V(m)(p_i)=\Sigma V(m_n)(p_i)=|a_i|$, it should be $K=1$, but $\sup V(m_n)(x) = \sup|a_n|$ while $V(m)(x) = \Sigma V(m_n)(x)=\Sigma|a_n|>\sup|a_n|$ (if there are two numbers nonzero al least).

## References.

[1] BATLE, N.:"Una nota sobre la construcción de medidas vectoriales". Actas de la XI RAME, Publ. CSIC (183-188), 197.

[2] DUNFORD, N.- SCHWARTZ, J. T.: "Linear Operators, Part I". Interscience. New York, 1967.

[3] GOWURIN, M.: "Uber die Stieltjesche Integration abstrakter Funktionen". Fundamenta Math. 27 (254-268), 1936.

[4] MARTIN, J. T.: "Introduction to the theory of bases". Springer Verlag, 1969.

Dept. Matemàtiques i Estadística,
E.T.S. d'Arquitectura,
Universitat Politècnica de Barcelona,
Av. Diagonal, 649,
BARCELONA-28. SPAIN.