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ON A GENERALIZATION OF SUM FORM FUNCTIONAL EQUATION-I (*)

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ABSTRACT

The Shannon entropy has the sum form $\Sigma f(p_i)$ *with* $f(x)=-x\log x$ $(x\in[0,1])$. *This together with the property of additivity lead to the 'sum' functional equation*

$$(1)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i y_j)=\sum_{i=1}^{m} f(x_i)+\sum_{j=1}^{n} f(y_j).$$

For more information about (1), refer to [2,1,4,6].

In this paper we determine measurable solution of the functional equation (which is a generalization of (1))

$$(2)\qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(x_i y_j)=\sum_{j=1}^{n} g_j(y_j)+\sum_{j=1}^{n} y_j^{\beta}\cdot\sum_{i=1}^{m} h_i(x_i).$$

1. Auxiliary Results.

Let $\Delta_n=\{X=(x_1,\ldots,x_n):\ x_i\geq 0,\ \sum_{i=1}^{n} x_i=1\}$. Before treating the functional equation (2), we first study the following functional equation which we will use several times:

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(3) $$\sum_{i=1}^{n} f_i(x_i) = c, \text{ with } X = (x_1,\dots,x_n) \in \Delta_n,$$

where f_i: $[0,1] \to R$(reals) and c is a constant.

First, it is easy to see that if any one of the functions in (3) is Lebesgue measurable, then so are the others. In the sequel, we assume that f_1 is measurable.

Lemma 1. Let f_i:$[0,1] \to R$ ($i = 1,2,3$) satisfy (3) with measurable f_1. Then

(4) $$f_i(x) = ax + b_i, \quad i = 1,2,3$$

where a,b_1,b_2,b_3 are constants with $a + \sum_{i=1}^{3} b_i = c$.

Proof. For n = 3, (3) takes the form

(5) $$f_1(u) + f_2(v) + f_3(1-u-v) = c, \text{ for } u,v,u+v\in[0,1],$$

which is a Pexider equation. Now, from the measurability of the f_i's ($i = 1,2,3$), it follows that f_i's have the form (4).

Remark 1. It is easy to see that all f_i's ($i = 1,2,\dots,n$) satisfying (3) have the form (4) with $a + \sum_1^n b_i = c$.

Now, we will prove another result whcih will be helpful in the next section.

Lemma 2. Let f,g,h,k: $[0,1] \to R$ be measurable and satisfy the functional equation

(6) $$f(xy)-g((1-x)y)-h(y)=y^{\beta}k(x), \text{ for } x,y\in[0,1], \ (\beta\neq 0)$$

with the convention $0^{\beta} = 0$. Then

(7) $$\begin{cases} k(x) = A[x^{\beta}+(1-x)^{\beta}-1] + Bx^{\beta}+ C(1-x)^{\beta}, \\ f(x) = (A+B)x^{\beta}+ Dx + D_1, \\ g(x) = -(A+C)x^{\beta}-Dx + D_1 - D_2, \\ h(x) = Ax^{\beta}+ Dx + D_2, \end{cases} \quad \text{for } \beta \neq 1$$

and

(8) $$\begin{cases} k(x) = A[x \log x + (1-x)\log(1-x)] + (B-C)x + C, \\ f(x) = Ax \log x + (B+D)x + D_1, \\ g(x) = -Ax \log x - (D+C)x + D_1 - D_2, \\ h(x) = Ax \log x + Dx + D_2, \end{cases} \quad \text{for } \beta = 1,$$

where A,B,C,D,D_i are arbitrary constants.

Proof. With $x = 0$ and $x = 1$, (6) becomes

(9) $$g(0) - g(y) - h(y) = y^{\beta}k(0)$$

(10) $$f(y) - g(0) - h(y) = y^{\beta}k(1).$$

Remark 2. From (9) and (10), we can conclude that if any one of f,g,h in (6) is measurable, so are the others and k. So, for example, Lemma 2 remains valid if we assume only that f in (6) is measurable. From (6) and (9) results

(11) $$f_1(xy) + h((1-x)y) - h(y) = y^{\beta}k_1(x),$$

where

(12) $$f_1(x) = f(x) - f(0),\ k_1(x) = k(x) - C(1-x)^{\beta}, \text{ with } C=k(0).$$

Now use (11) to get

$$k_1(x)+(1-x)^\beta k_1(\frac{y}{1-x})=2^\beta(f_1(\frac{x}{2})+h(\frac{1-x}{2})-h(\frac{1}{2}))+2^\beta[f_1(\frac{y}{2})+h(\frac{1-x-y}{2})-h(\frac{1-x}{2})],$$

from which, since the right hand side is symmetric in x and y we conclude that k_1 satisfies the functional equation

(13) $$k_1(x) + (1-x)^\beta k_1(\frac{y}{1-x})=k_1(y)+(1-y)^\beta k_1(\frac{x}{1-y}).$$

From [3,7,10], it follows that

(14) $$k_1(x) = A[x^\beta+(1-x)^\beta-1] + Bx^\beta, \text{ for } \beta \neq 1,$$

and

(15) $$k_1(x) = A[x \log x+(1-x)\log(1-x)]+ Bx, \text{ for } \beta = 1,$$

that is,

(16) $$k(x) = A[x^\beta+(1-x)^\beta-1) + Bx^\beta+ C(1-x)^\beta, \text{ for } \beta \neq 1,$$

and

(17) $$k(x) = A[x \log x+(1-x)\log(1-x)] + (B-C)x+C, \text{ for } \beta=1.$$

<u>Remark 3.</u> The measurability of k is used only to obtain (15) (case $\beta = 1$) while (14) holds whether k is measurable or not [refer to 3,7,10].

Let us first treat the case $\beta \neq 1$. From (11), (12) and (14) result the Pexider equation

(18) $$F(xy) + H((1-x)y) = H(y), \text{ for } x,y\in[0,1]$$

with $F(x) = f(x) - (A+B)x^\beta - f(0)$, $H(x) = h(x) - Ax^\beta$. Since F and H are measurable, $F(x) = Dx$, $H(x) = Dx + D_2$ so that

(19) $$f(x) = (A+B)x^{\beta} + Dx + D_1,$$

(20) $$h(x) = Ax^{\beta} + Dx + D_2.$$

From (9) and (19), we get

(21) $$g(x) = -(A+C)x^{\beta} - Dx + D_1 - D_2.$$

This values of k,h,f,g given by (16), (19), (20) and (21) satisfy (6). Similarly, for the case $\beta = 1$ using (11), (12), (17) and (18) with $F(x) = f(x) - Ax\log x - Bx - f(0)$, $H(x) = h(x) - Ax\log x$, we obtain (8). This completes the proof of this lemma.

2. Solution of the functional equation (2).

The case m = 2, n = 3 seems to require a technique different from all the other cases $m,n \geqslant 3$. So, it is appropriate to consider it separately. First, we treat the case m = 2, n = 3 and solve the functional equation

(2.1) $$\sum_{i=1}^{2}\sum_{j=1}^{3} f_{ij}(x_iy_j) = \sum_{j=1}^{3} g_j(y_j) + \sum_{j=1}^{3} y_j^{\beta} \cdot \sum_{i=1}^{2} h_i(x_i),$$

for $X = (x_1,x_2)\in\Delta_2$, $Y = (y_1,y_2,y_3)\in\Delta_3$, $\beta \neq 0$, where h_i, $f_{ij},g_j:[0,1] \to R$ with measurable f_{ij},g_j $(i=1,2,\ j=1,2,3)$, and the convention $0^{\beta} = 0$.

For fixed x, by defining

(2.2) $$\alpha_j(y)=f_{1j}(xy)+f_{2j}((1-x)y)-g_j(y)-y^{\beta}(h_1(x)+h_2(1-x)), \text{ for}$$

$y\varepsilon[0,1]$,

$j = 1,2,3$, (2.1) can be reduced to the Pexider equation (5) with $c = 0$ and α_j measurable. Thus by Lemma 1, we have

$$(2.3) \qquad \alpha_j(y) = a(x)y + b_j(x), \quad j = 1,2,3,$$

with $a(x) + b_1(x) + b_2(x) + b_3(x) = 0$. From (2.2) and (2.3), by taking $y = 0$, it is easy to see that, $b_1(x)$, $b_2(x)$, $b_3(x)$, and hence $a(x)$ are constants, so that we have

$$(2.4) \qquad \alpha_j(y) = ay + b_j, \quad j = 1,2,3,$$

where a, b_1, b_2 are constants with $a_1 + b_1 + b_2 + b_3 = 0$. From (2.2) and (2.4) for $j = 1$, we obtain

$$(2.5) \qquad f_{11}(xy)+f_{21}((1-x)y)-g_1(y)-ay-b_1= y^{\beta}(h_1(x)+h_2(1-x)),$$

which by the use of Lemma 3, gives

$$(2.6) \qquad \begin{cases} f_{11}(x) = Ax\ \log x + (B+D_1)x + D_{11}, \\ f_{21}(x) = Ax\ \log x + (D_1+C)x + D_{21}, \\ g_1(x) = Ax\ \log x + (D_1-a)x + D_{11}+ D_{21} - b_1, \\ h_1(x) = h(x), \\ h_2(x) = -h(1-x) + A[x\log x+(1-x)\log(1-x)]-(B-C)x+B, \end{cases} \quad \text{for } \beta = 1$$

and

$$(2.7) \qquad \begin{cases} f_{11}(x) = (A+B)x^{\beta} + D_1 + D_{11}, \\ f_{21}(x) = (A+C)x^{\beta} + D_1 x + D_{21}, \\ g_1(x) = Ax^{\beta} + (D_1-a)x + D_{11} + D_{21} - b_1, \quad \text{for } \beta \neq 1 \\ h_1(x) = h(x), \end{cases}$$

where $h: [0,1] \to R$ is arbitrary.

Similarly using (2.2), and (2.4) for $j = 2$, and $j = 3$ and noting (2.6) and (2.7), we have

$$(2.8)\qquad \begin{cases} f_{1j}(x) = Ax \log x + (B+D_j)x + D_{1j}, \; j = 2,3, \\ f_{2j}(x) = Ax \log x + (D_j+C)x + D_{2j}, \; j = 2,3, \\ g_2\,(x) = Ax \log x + (D_2-a)x + D_{12} + D_{22} - b_2, \; \text{for } \beta=1 \\ g_3(x) = Ax \log x + (D_3-a)x + D_{13} + D_{23} + b_1 + b_2 + a. \end{cases}$$

and

$$(2.9)\qquad \begin{cases} f_{1j}(x) = (A+B)x^{\beta} + D_jx + D_{1j}, \; j = 2,3 \\ f_{2j}(x) = (A+C)x^{\beta} + D_jx + D_{2j}, \; j = 2,3 \\ g_2(x) = Ax^{\beta} + (D_2-a)x + D_{12} + D_{22} - b_2, \\ g_3(x) = Ax^{\beta} + (D_3-a)x + D_{13} + D_{23} + b_1 + b_2 + a. \end{cases} \quad \text{for } \beta \neq 1$$

Thus we have proved the following theorem.

<u>Theorem 1.</u> The most general 'measurable' solution of the functional equation (2.1) is given either by (2.6) and (2.8), or by (2.7) and (2.9) according as $\beta=1$ or $\beta\neq1$.

<u>Remark 4.</u> The equation (2.1) is a generalization to the functional equation considered in [9].

Lastly, we take up the case $m = 3 = n$ and solve the functional equation

$$(2.11)\qquad \sum_{i=1}^{3} \sum_{j=1}^{3} f_{ij}(x_iy_j) = \sum_{j=1}^{3} g_j(y_j) + \sum_{j=1}^{3} y_j^{\beta} \cdot \sum_{i=1}^{3} h_i(x_i),$$

for $X,Y \in \Delta_3$, $\beta \neq 0$ with $0^{\beta} = 0$, where $f_{ij}, g_j h_i$: $[0,1] \to R$ are measurable $(i,j=1,2,3)$. For fixed $X = (x_1,x_2,x_3) \in \Delta_3$, defining

$$(2.12)\quad \alpha_j(y)=f_{1j}(x_1y)+f_{2j}(x_2y)+f_{3j}(x_3y)-g_j(y)-y^\beta(h_1(x_1)+h_2(x_2)+h_3(x_3)),\ \text{for } y\in[0,1],\ j=1,2,3,$$

(2.11) takes the form of (5), with $c = 0$.

As before, we can show that,

$$(2.13)\qquad \alpha_j(y) = ay + b_j,\ j = 1,2,3$$

where a,b_i's are constants, with $a + b_1 + b_2 + b_3 = 0$.

Using (2.12) and (2.13) for $j = 1$, we get

$$(2.14)\quad f_{11}(uy)+f_{21}(vy)+f_{31}((1-u-v)y)=g_1(y)+y^\beta(h_1(u)+h_2(v)+h_3(1-u-v))+ay+b_1$$

for $y,u,v,u+v\in[0,1]$, which could be put into the Pexider equation $F_1(u) + F_2(v)=F_2(u+v)$ with $F_1(u)=f_{11}(uy)-y^\beta(h_1(u)-h_1(0))-f_{11}(0)$, $F_2(v) = f_{21}(vy) - y^\beta h_2(v)$, for $u,v\in[0,1]$

so that

$$(2.15)\qquad f_{21}(vy) = A(y)v + B(y) + y^\beta h_2(v),\ \text{for } v,y\in[0,1].$$

$$(2.16)\qquad f_{11}(uy) = A(y)u + y^\beta(h_1(u)-h_1(0))+f_{11}(0),\ \text{for } u,y\in[0,1].$$

From (2.16), using the symmetry in u and y of the left side, we obtain

$$(2.17)\qquad h_1(u) = A(u) + c_1u^\beta + c_2u + D_1,$$

$$(2.18)\qquad f_{11}(u)= A(u) + c_1u^\beta + D_{11},\ \text{for } u\in[0,1].$$

From (2.16), (2.17) and (2.18), we have

$$A(y)(u^\beta-u) = A(u)(y^\beta-y)+c_2(uy^\beta-u^\beta y).$$

Thus, if <u>$\beta \neq 1$</u>, for fixed u (say = 1/2), we get

$$(2.19) \qquad A(y) = c_4 y^\beta + c_5 y, \text{ for } y \in [0,1].$$

Hence, when $\beta \neq 1$, (2.16), (2.17) and (2.18) give

$$(2.20) \qquad \begin{cases} f_{11}(y) = A_1 y^\beta + E_1 y + D_{11}, \\ h_1(y) = A_1 y^\beta + B_1 y + D_1, \text{ for } y \in [0,1], \text{ for } \beta \neq 1. \end{cases}$$

Now f_{11}, h_1 and A given by (2.20) and (2.19) satisfy (2.16) provided $c_4 = -B_1$ and $c_5 = E_1$, that is,

$$(2.21) \qquad A(y) = -B_1 y^\beta + E_1 y.$$

By using similar and analogous arguments (using (2.15)), we obtain

$$(2.22) \qquad \begin{cases} f_{j1}(y) = A_j y^\beta + E_1 y + D_{j1}, \ j = 2,3 \\ h_j(y) = A_j y^\beta + B_1 y + D_j, \ j = 2,3, \text{ for } \beta \neq 1, \end{cases}$$

so that from (2.14), (2.20) and (2.22), we have

$$(2.23) \qquad g_1(y) = -(B_1+D_1+D_2+D_3)y^\beta + (E_1-a)y+D_{11}+D_{21}+D_{31}-b_1.$$

Similarly, when $\beta \neq 1$, we have

$$(2.24) \qquad \begin{cases} f_{ij}(y) = A_i y^\beta + E_j y + D_{ij}, \ i = 1,2,3, \ j = 2,3, \\ g_2(y) = -(B_1 + \sum_{i=1}^{3} D_i) y^\beta + (E_2-a)y + \sum_{i=1}^{3} D_{i2} - b_2, \\ g_3(y) = -(B_1 + \sum_{i=1}^{3} D_1) y^\beta + (E_3-a)y + \sum_{i=1}^{3} D_{i3}+b_1+b_2+a. \end{cases}$$

Thus, (2.20), (2.22), (2.23) and (2.24) constitute the general measurable solution of (2.11), when $\beta \neq 1$.

Finally, we consider the case $\underline{\beta = 1.}$ Now, from (2.16), (2.17) and (2.18), we get

$$A(uy) = A(u)y + A(y)u + c_2 uy, \text{ for } u,y \in [0,1],$$

so that

$$(2.25) \qquad A(y) = Ay \log y - c_2 y, \text{ for } y \in [0,1].$$

Now, (2.25) and (2.18) with $\beta = 1$ give

$$(2.26) \qquad \begin{cases} f_{11}(y) = Ay \log y + B_{11}y + D_{11}, \\ h_1(y) = Ay \log y + B_1 y + D_1, \text{ for } y \in [0,1] \end{cases}$$

where $A, B_{11}, B_1, D_{11}, D_1$ are constants.

The functions f_{11}, h_1 and A given by (2.26) and (2.25) satisfy (2.16) with $\beta = 1$, provided $c_2 = B_1 - B_{11}$, that is,

$$(2.27) \qquad A(y) = Ay \log y + (B_{11} - B_1)y.$$

As before, be analogous arguments, we obtain

$$(2.28) \qquad \begin{cases} f_{i1}(y) = Ay \log y + B_{i1} + D_{i1}, \quad i = 2,3 \\ h_i(y) = Ay \log y + B_i y + D_i, \quad i = 2,3, \text{ for } y \in [0,1] \\ g_1(y) = Ay \log y + (B_{i1} - B_i - \sum_{j=1}^{3} D_j - a)y + \sum_{k=1}^{3} D_{k1} - b_1, i=1 \text{ or } 2 \\ \qquad\qquad \text{or } 3, \end{cases}$$

with $B_{11} - B_1 = B_{21} - B_2 = B_{31} - B_3$.

Similarly, when $\beta = 1$, we get

$$(2.29)\quad \begin{cases} f_{ij}(y)=Ay\log y+B_{ij}y+D_{ij},\ i=1,2,3,\ j=2,3 \\ g_2(y)=Ay\log y+(B_{i2}-B_i-\sum_{j=1}^{3} D_j-a)y+\sum_{k=1}^{3} D_{k2}-b_2, \text{ for } i=1 \text{ or } 2 \text{ or } 3. \\ g_3(y)=Ay\log y+(B_{i3}-B_i-\sum_{j=1}^{3} D_j-a)y+\sum_{k=1}^{3} D_{k3}+b_1+b_2+a \text{ for } i=1 \text{ or } 2 \text{ or } 3, \end{cases}$$

with $B_{12}-B_1=B_{22}-B_2=B_{32}-B_3$, $B_{13}-B_1=B_{23}-B_2=B_{33}-B_3$. Thus, (2.26), (2.20), (2.28) and (2.29) form the general measurable solution of (2.11), when $\beta = 1$. Thus, we have proved the following theorem.

<u>Theorem 2.</u> Let $f_{ij}, h_i, g_j : [0,1] \to R$ be measurable $(i,j=1,2,3)$ and satisfy the functional equation (2.11). Then the solutions of (2.11) are given either by (2.20),(2.22), (2.23) and (2.24) or by (2.26), (2.28), and (2.33) according as $\beta \neq 1$ or $\beta = 1$.

<u>Remark 5.</u> The measurable solution if the function equation (2) holding for all m and n or for any particular pair $m,n \geqslant 3$ can be obtained by adopting the method in Theorem 2.

<u>Remark 6.</u> The continuous solutions of all the functional equations treated in [8] can be obtained easily from the results in sections 2 and 3.

3. Applications.

When $\beta = 1$ and $f_{ij} = g_j = h_i = f$, (2) reduces to (1), the subject of study in [2,1,4,6]. Thus when $\beta = 1$ and all the functions involved in (2.1) or (2.11) are equal to f (measurable), then

$$f(x) = Ax\log x + Bx + B \text{ and } f(x) = Ax\log x + 3Bx + B, \text{ for } x\in[0,1]$$

with $0 \log 0 = 0$ are the solutions of (2.1) and (2.11) respectively, where A and B are arbitrary constants; that is, the $\beta = 1$ case leads to a characterization of Shannon's entropy.

When $\beta \neq 1$ ($\neq 0$) and $f_{ij} = g_j = h_i = f$, (2) reduces to the functional equation

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{j=1}^{n} f(y_j) + \sum_{j=1}^{n} y_j^{\beta} \cdot \sum_{i=1}^{m} f(x_i).$$

Thus when $\beta \neq 1$ ($\neq 0$) and all the functions occurring in (2.1) or (2.11) are equal to f (measurable), then

$$f(x) = A(x^{\beta} - x), \; x \in [0,1],$$

where A is an arbitrary constant; that is, $\beta \neq 1$ case leads to a cgaracterization of the entropy of type β investigated in [5,3] .

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$$\sum_{i=1}^{2} \sum_{j=1}^{3} F_{ij}(p_i q_j) = \sum_{i=1}^{2} G_i(p_i) + \sum_{j=1}^{3} H_j(q_j),$$

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