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STOCHASTIC SIGNAL CODIFICATION AND SIGMA TRANSFORM

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ABSTRACT

In the last years, a relation between bounded real functions of one variable and two-valued probabilistic functions defined on the complex plane has been established through the introduction of the Sigma-Transform concept.

The paper presents an extension of the concept of Sigma-Transform, giving rise to the Diagonal Sigma-Transform and the Striped Sigma-Transform which, when combined, allow a formal treatment of the Multichannel Stochastic Signal Codification. An application of the method to error analysis is presented.

1. Introduction.

In stochastic signal codification, time variables are represented by means of the probability of pulse ocurrence of random pulse trains. [5, 7, 1]. This method of signal coding allows, among other properties, the processing of these signals by means of simple logic operations. [6, 2].

The interdependence between logic and arithmetic operations made apparent by stochastic signal condification. suggested the development of a more general theory, connecting both kinds of operations by means of a mathematical transform. This transform, which has been called Sigma-Transform, [4] relates bounded real functions with two-valued probabilistic functions defined on the complex plane.

The direct transform converts each value of the real function into an infinite set of two-valued states distributed along a direction parallel to the imaginary axis, the state probability being the real function value. [3].

In order to formalize the stochastic signal codification, two special kinds of Sigma-Transforms are defined: the Diagonal Sigma-Transform, in which the state values of two consecutive time instants are equal almost everywhere, and the Striped Sigma-Transform where the sequences of state values along the imaginary axis are preiodic.

Sigma-Transforms which are at the same time Diagonal and Striped, will be called Diastriped Sigma-Transforms, and allow a formal description of stochastic signal codification.

2. Σ-Transform of a function.

Let x be a real function $A \to R$. If $A=R$, x will represent a continuous-time signal, and if $A=Z$, a discrete-time or sampled signal.

A Σ-Transform of x is a function

$$X = \Sigma[x] : A \times N \to \{0,1\}$$

such that

$$\lim_m \frac{1}{m} \sum_{n=1}^{m} X(a,n) = \Sigma^{-1}[X(a,n)] = x(a)$$

for every $a \in A$.

The set A x N can be considered imbeded in the complex plane C if A is contained in R and then X(a,n) becomes x(a+jn).

In [3], the existance of a Σ-Transform for every function x(a) and its generation via random noise has been shown.

In this paper the case A=N, corresponding to sampled signals will only be considered.

3. Diagonal Σ-Transform.

Let be $x: Z \to R$ a constant function $x(m) = \alpha$ for every n belonging to Z and let be $\{a_i\}$ a sequence of zeroes and ones such that

$$\lim_n \frac{1}{n}(a_1 + \ldots + a_n) = \alpha$$

Take $X(1,1)=a_1$, $X(1,2)=a_2, \ldots, X(1,n)=a_n \ldots$ and define by induction $X(m+1,i)$ $i=1,2,\ldots$ from $X(m,i)$ by taking $X(m+1,1)$ arbitrary and $X(m+1,2)=X(m,1) \ldots X(m+1,k)=X(m,k-1)$. (In fact $X(k,j)$ is constant on the subsets of Z x N given by k-j=constant).

The function X(k,n) is a Σ-Transform of the constant function x(m): if $\lim_n \frac{X(m,1)+\ldots+X(m,n)}{n} = \alpha$

one verifies that

$$\lim_n \frac{1}{n}(X(m+1,1)+\ldots+X(m+1,n) =$$

$$= \lim_n \frac{X(m,1)+\ldots+X(m,n)}{n} + \lim_n \frac{1}{n}[X(m+1,1)-X(m,n)]$$

Conversely, if the function X(m,n) define as above is a

Σ-Transform of a function $x(m)$, this function $x(t)$ must be a constant.

The Σ-Transform $X(m,n)$ of functions $x(m)$ such that $X(m,n) = X(m-1,n-k)$ will be called Diagonal Σ-Transform.

The former results can be stated as follows:

Proposition. The function $x(m)$ has a Diagonal Σ-Transform if and only if $x(m)$ is constant.

The above result can also be interpreted in the sense that it is not possible to represent sampled time-varying signals by means of an infinite sequence of two-valued states at each time instant when there is only a finite number of state changes between two succesive sequences.

4. Striped Σ-Transform.

Let be $x : Z \to R$ a function and ε a positive real number. A function $X(m,n)$ is called approximated Σ-Transform of $x(m)$ with error ε if

$$\left| \lim_n \frac{1}{n} (X(m,1)+\ldots+X(m,n))-x(m) \right| < \varepsilon$$

It is obvious that given $\varepsilon>0$ there always exists approximated Σ-Transforms with error ε. It suffices to take an ordinary (exact) Σ-Transform.

Proposition. Given $x : Z \to R$ and $\varepsilon>0$ there exists an approximated Σ-Transform with error ε which is periodic, that is, such that the sequences

$$\{X(m,n)\}_n \quad \text{are periodic for every } m$$

That transform will be called Striped Σ-Transform.

Proof: Given $\varepsilon>0$, let $X(m,n)$ be a Σ-transform of $x(m)$. For $n>n_o(m)$ we have

$$\left| \frac{1}{n} [X(m,1)+\ldots+X(m,n)] - x(m) \right| < \varepsilon/2$$

Consider the periodic sequence obtained by repetition of the $n_o(m)$ numbers

$$X(m,1),\ldots,X[m,n_o(m)]$$

The above proposition comes from the

<u>Lemma.</u> If $\{a_i\}_i$ is a periodic sequence with fundamental period $a_1\ a_2\ldots a_K$, then

$$\lim_i \frac{a_1+\ldots+a_i}{i} = \frac{a_1+\ldots+a_K}{K}$$

Indeed, let $\alpha = \frac{a_1+\ldots+a_K}{K}$ and put $S_i = \frac{a_1+\ldots+a_i}{i}$. By integer division we obtain $i = Kr+h \quad 0 \leqslant h < K$ and

$$S_i = \frac{rK\alpha+a_1+\ldots+a_h}{Kr+h}$$

$$|S_i-\alpha| \leqslant \frac{|a_1-\alpha|+\ldots+|a_h-\alpha|}{rK+h} \leqslant \frac{h}{rK+h} < \frac{1}{r} < \varepsilon/2 \text{ if } r>\frac{2}{\varepsilon}$$

5. Diastriped Σ-Transform and Stochastic Signal Codification.

Let be $x(n)$ a function $x:Z \to [0,1]$ and k, p two positive integers and $P = \{1,2,\ldots,kp\}$, $P_1 = \{1,\ldots,p\}$.

If $X(m,n)$, $m\epsilon P_1$ is a random variable which takes only the values 0 and 1, obtained as in [3], we have

$$E[X(m,n)] = x(m)$$

$$Var[X(m,n)] = [1-x(m)]\, x(m) \quad (\leqslant 1/4)$$

Define the new random variables

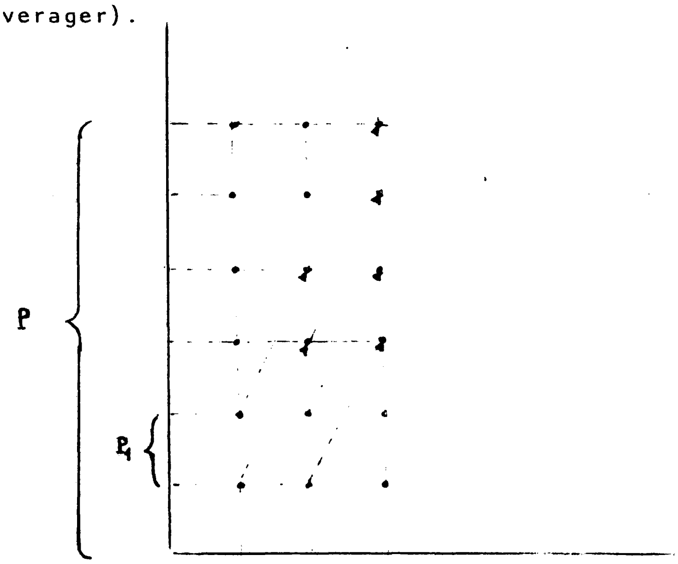
$$X(m, rp+i) = X(m-r, i)$$

so that

$$E[X(m,rp+i)] = x(m-r)$$

$$Var[X(m,rp+i)] = x(m-r)\,[1-x(m-r)] \quad (\leqslant 1/4)$$

The above expressions allow the substitution of an average along the whole of a vertical axis corresponding to a Diastriped Sigma-Transform, by the average of a finite number of states along the vertical axes related to several consecutive time instants. In other words, the Inverse Diastriped Σ-Trasnform is equivalent to the recovery of a multichannel stochastic codificated signal, as obtained from a uniformly weighted bidimensional averager (flat averager).



Plane of a Diastriped Σ-Transform with p=2, k=3.

5.1. Error analysis.

Given a time function x(t) the Σ-Transform can always be obtained. From this Σ-Transform, once the k and p parameters are fixed, the corresponding Diastriped Σ-Transform can be constructed. The Inverse Diastriped Σ-Transform and the original time signal do not exactly coincide. The difference is due to errors of two types: statistical and dynamical.

The considerations of the previous point justify the translation of error analysis of Diastriped Σ-Transforms to stochastic multichannel signal codification.

The variances of the random variables X(m,n) are bounded and weak law of large numbers gives

$$\text{Prob}\left(\left|\frac{\sum_{i\in P} X(m,i)}{pk} - \frac{px(m)+\ldots+px(m-k+1)}{pk}\right| \geqslant \varepsilon\right) <$$

$$< \frac{1}{\varepsilon^2 p^2 k^2} \sum_{i\in P} \text{Var } X(m,i) < \frac{1}{4\varepsilon^2 pk}$$

so that

$$\left|\frac{\sum_{i\in P} X(m,i)}{pk} - \frac{x(m)+\ldots+x(m-k+1)}{k}\right| < \varepsilon$$

In order to determine the dynamical errors through the comparation of $\frac{x(m)+\ldots+(m-k+1)}{k}$ and x(m), restrictive hypothesis about x(m) must be made.

5.1.1. Bounded Jump Sampled Signals.

Let x(m) be a discrete-time bounded jump function, that is $|x(m) - x(m+1)| < \beta$, for every $m \in Z$, and consequently $|x(m)-x(m+k)| < k\beta$.

Lemma. If $a_1, a_2 \ldots a_{2k+1}$ are real numbers such that $|a_i - a_{i+1}| < \beta$, then the bound of

$$\left|\frac{a_1+\ldots+a_{2k+1}}{2k+1} - a_i\right|$$

has a minimum for $i=k+1$ and the bound is

$$\beta\,\frac{k(k+1)}{2k+1}$$

<u>Proposition.</u> Let $x(m)$ be a discrete-time signal of bounded jump β and let k be such that

$$\beta\,\frac{k(k+1)}{2k+1} < \varepsilon/2$$

Then there exists a Diastriped Σ-Transform of $y(m)=x(m-k-1)$ with parameters p and $2k+1$, and such that the probability of obtaining a realization via random noise is greater than

$$1 - \frac{1}{\varepsilon^2 p(2k+1)}$$

By the lemma we have

$$\left|y(m) - \frac{y(m)+y(m-1)+\ldots+y(m-2k)}{2k+1}\right| < \varepsilon/2$$

and the proposition follows from

$$\text{Prob}\left(\left|\frac{\sum_{i\in P} X(m,i)}{p(2k+1)} - \frac{x(m)+\ldots+x(m-2k)}{2k+1}\right| < \varepsilon/2\right) >$$

$$> 1 - \frac{1}{\varepsilon^2 p(2k+1)}$$

5.1.2. Band-limited Signals.

Let $x(m)$ be a function $x: Z \to R$. It is known that there exists a biyective correspondence between such functions and $\mathcal{L}^2(-\infty,+\infty)$

(functions of finite energy) which are band limited to $[-\pi,\pi]$. (Sampling theorem).

The operator $Z^R \to Z^R$ given by

$$x(m) \to y(m) = \frac{1}{k}\,(x(m)+\ldots+x(m-k+1))$$

has an equivalent operator $\mathcal{L}^2(-\infty,+\infty) \to \mathcal{L}^2(-\infty,+\infty)$ such that, when extended to the distributions over R, transforms δ_o into $\frac{1}{k}\,u(t)\;u(k-t)=h(t)$.

This operator is

$$x(t) \to y(t) = x(t) * h(t) \quad \text{(vonvolution product)}$$

By application of Fourier transform, we have

$$Y(\omega) = X(\omega)\cdot H(\omega)$$

with

$$H(\omega) = \frac{2\sin k\omega}{k\;\omega}e^{-j\omega\frac{k}{2}}$$

If $z(t) = x(t-\frac{k}{2})$, we can write

$$Y(\omega) - Z(\omega) = X(\omega)\;H(\omega) - X(\omega)\;e^{-j\omega\frac{k}{2}} =$$

$$= X(\omega)\;e^{-j\omega\frac{k}{2}}\;\left(\frac{\sin\frac{k}{2}\omega}{\frac{k}{2}\omega} - 1\right)$$

and we obtain

$$y(t)-x(t-\frac{k}{2}) = \frac{1}{2\pi}\int_{-\sigma}^{+\sigma} X(\omega)\;e^{-j\omega\frac{k}{2}}\;\left(\frac{\sin\frac{k}{2}\omega}{\frac{k}{2}\omega} - 1\right)\,d\omega$$

where σ is a bound for the band of $x(t)$. $(0 \leqslant \sigma < \pi)$. Finally we can write

$$\left|y(t) - x(t-\frac{k}{2})\right| \leqslant \frac{\sigma}{\pi}\;\left|X(\omega)\right|\;(1-\frac{\sigma\frac{k}{2}\;\omega}{\frac{k}{2}\;\omega})$$

Proposition. Let x(m) be the discrete-time function $x: Z \to R$ associated to a continuous-time function x(t) of limited band in $[-\sigma,\sigma]$, and let K be a bound of $|X(\omega)|$, the Fourier transform of x(t). If

$$\frac{\sigma}{\pi} K \left(1 - \frac{\sin \frac{k\sigma}{2}}{\frac{k\sigma}{2}}\right) < \varepsilon/2$$

then there exists a Diastriped Σ-Transform of the function $x(t-\frac{k}{2})$, and the probability of obtaining a realization by means of random noises is greater than

$$1 - \frac{1}{\varepsilon^2 p\ k}$$

6. Conclusions.

The introduction of the Diastriped Σ-Transform concept supplies a very adequate tool for the formal treatment of the multichannel stochastic signal codification phenomena. The method shows also big potentialities if extended to the study of stochastic multichannel signal processing.

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