STOCHASTICA Vol. VII, nº 1 (1983)

A CLOSURE CONDITION WHICH IS EQUIVALENT TO THE THOMSEN CONDITION IN QUASIGROUPS

M. A. Taylor

ABSTRACT

In this note it is shown that the closure condition, $X_1Y_2=X_2Y_1$, $X_1Y_4=X_2Y_3$, $X_3Y_3=X_4Y_1 \to X_4Y_2=X_3Y_4$, *(and its dual) is equivalent to the Thomsen condition in quasigroups but not in general. Conditions are also given under which groupoids satisfying it are principal homotopes of cancellative, abelian semigroups, or abelian groups.*

The relationships between the Reidemeister condition and group isotopes and the Thomsen condition and abelian group isotopes are well known. In [1] groupoids satisfying one or both of these closure conditions are investigated. The closure condition (and its dual) presented here is equivalent to Thomsen for quasigroups and certain other groupoids but not in general.

This closure condition has been used to solve certain classes of functional equations for which the Thomsen condition presented technical difficulties [2].

<u>Definitions:</u> The <u>Thomsen</u> condition is said to hold in a groupoid $(G,\cdot)$ if for all X_i, $Y_i \varepsilon G$ $(i = 1,2,3)$

$$X_1Y_2 = X_2Y_1 \text{ and } X_1Y_3 = X_3Y_1$$

imply

$$X_2Y_3 = X_3Y_2.$$

Closure condition <u>A</u> is said to hold in $(G,\cdot)$ if for all X_i, Y_i

$(i = 1,2,3,4)$ $X_1Y_2 = X_2Y_1$ and $X_1Y_4 = X_2Y_3$ and

$$X_3Y_3 = X_4Y_1 \text{ imply } X_4Y_2 = X_3Y_4.$$

The dual of A, <u>B</u>, is given by

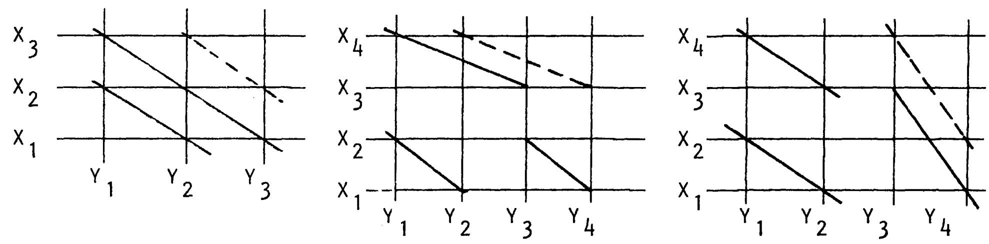
$$X_1Y_2 = X_2Y_1 \text{ and } X_4Y_1 = X_3Y_2 \text{ and } X_3Y_3 = X_1Y_4$$

imply $X_2Y_4 = X_4Y_3$.

A groupoid $(G,\cdot)$ is <u>right cancellative equivalent, r.c.e.,</u> if for all X_1, X_2, $Y_1 \varepsilon$ G

$X_1Y_1 = X_2Y_1$ implies $X_1Y = X_2Y$ for all $Y \varepsilon G$. <u>Left cancellative equivalent, l.c.e.</u>, is similarly defined and a groupoid is said to be <u>cancellative equivalent</u> if it is both left and right cancellative equivalent.

Illustrations of the closure conditions:



<u>Thomsen</u> <u>A</u> <u>B</u>

A solid line through the intersection of X_iY_j and X_kY_1 represents the equation $X_iY_j = X_kY_1$; a broken line represents the implied equation.

<u>Basic Results:</u>

<u>Lemma.</u> A groupoid, $(G,\cdot)$, in which A (B) holds is l.c.e. (r.c.e.)

Proof: Suppose $x_1y_1 = x_1y_2$, y_1, y_1, $y_2 \in G$. Let $x \in G$. Then the hypothesis of A is satisfied with $Y_1 = Y_2 = Y_3 = y_1$, $Y_4 = y_2$, $X_1 = X_2 = x_1$, $X_3 = X_4 = x$. The implication is that $xy_1 = xy_2$.

The dual form is similarly obtained.

It does not follow that a groupoid satisfying A is necessarily r.c.e. Let S be a non empty set for which there exists a bijection $f: SxS \to S$, and let x_1, x_2, $y_1 \in S$. Define $(S,\cdot)$ by

$$st = f(s,t) \qquad s \neq x_2,\ t \neq y_1$$

and

$$x_2y_1 = x_1y_1.$$

Then this groupoid satisfies A but not B. The dual of this groupoid satisfies B but not A, and T holds in both groupoids. This demostrates that A, B and T are not in general equivalent.

<u>Theorem 1.</u> The closure conditions A and T are equivalent in groupoids $(G,\cdot)$ for which there exists $a \in G$ such that $a\cdot G = G\cdot G$.

Proof: Let $(G,\cdot)$ be a groupoid and let $a \in G$ satisfy $a\cdot G = G\cdot G$.

Suppose that the closure condition A holds for the groupoid.

Assume that $x_1y_2 = x_2y_1$, $x_1y_3 = x_3y_1$, x_i, $y_i \in G$ $(1=1,2,3.)$

The groupoid's properties ensure there exist $y_4, y_5, y_6 \in G$

such that $x_1y_2 = ay_4$, $x_3y_2 = ay_5$ and $x_1y_5 = ay_6$.

The equations

$$ay_4 = x_1y_2,\ ay_6 = x_1y_5 \text{ and } ay_5 = x_3y_2$$

satisfy the hypothesis of A in a degenerate form, and the implication is $x_3y_4 = ay_6$. This leads to the equation, $x_3y_4 = x_1y_5$.

The hypothesis of A is now satisfied by

$$x_3y_4 = x_1y_5,\ x_3y_1 = x_1y_3,\ x_2y_1 = ay_4$$

with the conclusion that

$$x_2y_3 = ay_5.$$

Consequently, $x_2y_3 = x_3y_2$ and T is thereby satisfied in the groupoid.

For the converse, suppose that T holds in (G.·) and assume $x_1y_2 = x_2y_1$, $x_1y_4 = x_2y_3$ and $x_3y_3 = x_4y_1$. Let $y_5, y_6, y_7, y_8 \varepsilon G$ be such that

$$x_1y_2 = ay_5,\ x_1y_4 = ay_6,\ x_3y_3 = ay_7,\ x_3y_4 = ay_8$$

Then $x_2y_3 = ay_6$ and $x_3y_3 = ay_7$ imply $x_3y_6 = x_2y_7$; $x_2y_1 = ay_5$ and $x_4y_1 = ay_7$ imply $x_4y_5 = x_2y_7$; $x_1y_4 = ay_6$ and $x_3y_4 = ay_8$ imply $x_3y_6 = x_1y_8$. As a consequence of these three closures, $x_4y_5 = x_1y_8$.

This equation together with $x_1y_2 = ay_5$ gives the closure $x_4y_2 = ay_8$.

The required result that $x_4y_2 = x_3y_4$ is then immediate.

The dual of the above yields,

<u>Theorem 2.</u> The closure conditions B and T are equivalent in groupoids (G,·) for which there exists $b \varepsilon G$ such that $G \cdot b = G \cdot G$.

Corollary. The closure conditions A and B are equivalent to the Thomsen condition in quasigroups.

The Main Theorems.

Theorem 3. Let $(G,\cdot)$ be a groupoid in which A or B hold. If there exist elements, $a, b \in G$ such that $a\cdot G = G\cdot b = G\cdot G = S$ then the operation $*$ given by

$$(xb) * (ay) = xy \quad \text{for all } x, y \in G$$

is a cancellative, abelian semigroup operation. Moreover, $(S,*)$ has a unit element.

Proof: The lemma shows that $(G,\cdot)$ is cancellative equivalent and this ensures that the operation $*$ is well defined.

It is easily checked that ab is a unit element for $(S,*)$.

That T holds in $(G,\cdot)$ is a consequence of theorem 1 and it is easily shown that $(S,*)$ inherits T from $(G,\cdot)$. The theorem is completed by employing theorem 3 of [1] which states that a groupoid with a unit element is a cancellative abelian semigroup if and only if it has T.

The final theorem is analogous to theorem 5 of [1].

Theorem 4. Let $(G,\cdot)$ be a groupoid in which A or B hold. If there exist $a, c \in G$ such that $a\cdot G = G\cdot G = S$ and for all $y \in G$ there exist $x \in G$ satisfying $xy = c$, then for all $u, v \in G$ the operation $*$ defined by

$$(xv) * (uy) = xy \quad \text{for all } x, y \in G,$$

is an abelian group operation.

Proof: A similar theorem is proven in [1] for groupoids satisfying T. In view of this, and theorem 1 all that remains is th prove that if B holds in $(G,\cdot)$ then T also holds. This will be achieved by showing that there exists $b \in G$ such that $G\cdot b = G\cdot G$.

Let b be any member of G which satisfies $ab = c$. Now let y be an arbitrary member of G and consequently ay is arbitrary in S. It is then possible to find $x_1 \in G$ to satisfy $x_1 y \doteq c = ab$.

There exists $y_1 \in G$ such that $x_1 b = ay_1$ and in turn there is $x_2 \in G$ satisfying $x_2 y_1 = ab$.

This establishes a set of equations which constitute the hypothesis of B in a degenerate form. The conclusion is that $x_2 b = ay$.

It follows immediately that $G\cdot b \supset a\cdot G$ and therefore, $G\cdot b = S$.

This work was supported in part by N.S.E.R.C. Grant # A8144.

References

[1] TAYLOR, M. A., R-and-T-Groupoids: A generalization of Groups. Aequations Math. 12 242-248 (1975).

[2] TAYLOR, M. A., On a Problem of Krapež, Proceedings of the Symposium n-ary Structures, Macedonian Academy of Sciences and Arts, Skopje, 1982.

Acadia University
Wolfville, Nova Scotia
Canada BOP 1XO.