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# INTERIOR IDEMPOTENTS AND NON-REPRESENTABILITY OF GROUPOIDS

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Let $T: I \times I \to I$ be a groupoid on $I$ where $I = [a,b]$ is a subinterval of the extended real line. It is of interest to determine conditions under which T admits a representation of the form

$$T(x,y) = f(g(x)+h(y)), \tag{1}$$

in terms of continuous functions f, g and h.

One such set of conditions is contained in the following result of J. Aczel [1] and C. H. Ling [3]. Let T be an Archimedean semigroup on I, i.e. let T be an associative function such that

a) T is continuous on $I \times I$;

b) T is non-decreasing in each place;

c) The endpoint b of I is a unit, i.e., $T(b,x) = T(x,b) = x$ for all x;

d) T has no interior idempotents; i.e., $T(x,x) \neq x$ for all x in $(a,b)$.

Then T can be represented in the following strengthened form of (1):

$$T(x,y) = f(g(x)+g(y)) \tag{2}$$

where f and g are continuous, monotonic, and, in a certain well-defined sense, inverses.

Condition (d) is essential. Consider the function

(3) $$T(x,y) = \mathrm{Min}(x,y) \text{ for } x,y \text{ in } I,$$

which satisfies conditions (a)-(c) but violates condition (d) at all points of I. Not only is T non-representable in the form (2), but, as V.I. Arnold and A.A. Kirilov [2] have pointed out, T does not even admit representation (1).

The Min function is not a completely satisfactory counter-example, since it violates condition (d) to such an extreme degree. It is the purpose of this note to prove the following generalization of the Arnold-Kirilov result, which shows that if condition (d) fails at even a single point, then representation (1) does not hold.

Theorem. Let $I = [a,b]$ be a subinterval of the extended real line, with $T: I \times I \to I$ such that

(4) $T(b,b) = b$;

(5) $T(x,a) = T(a,x) = x$ for all x in I;

(6) the functions $\phi(x) = T(b,x)$, $\psi(x) = T(x,b)$ are continuous on I;

(7) the functions $\phi(x)$, $\psi(x)$ are idempotent, i.e. $\phi(\phi(x)) = \phi(x)$, $\psi(\psi(x)) = \psi(x)$ for all x in I.

If $T(\bar{x},\bar{x}) = \bar{x}$ for some $\bar{x}$ in (a,b), then T admits no representation of the form

(1) $$T(x,y) = f(g(x) + h(y))$$

where f, g and h are continuous functions.

Proof. Suppose continuous functions f, g, h satisfying (1) exist. Functions $\eta$, $\nu$ are defined on I as follows:

$$\eta(x) = g(b) + h(x)$$
$$\nu(x) = g(x) + h(b),$$

so that

$$(f\circ\eta)x = T(b,x) = \phi(x)$$

and

$$(f\circ\nu)x = T(x,b) = \psi(x).$$

Both $f\circ\eta$ and $f\circ\nu$ are continuous by (6), and idempotent by (7).

The range of $f\circ\eta$ includes $(f\circ\eta)a = T(b,a) = a$ and $(f\circ\eta)b = T(b,b) = b$. By the continuity of $f\circ\eta$, its range contains $[a,b]$. Similarly, the range of $f\circ\nu$ contains $[a,b]$. An idempotent function restricted to its range is the identity function; thus

(8) $$(f\circ\eta)x = T(b,x) = x,$$

(9) $$(f\circ\nu)x = T(x,b) = x \quad \text{for all } x \text{ in } I.$$

Hence the functions $\eta$, $\nu$ are one-to-one. It follows that the functions g, h are one-to-one; by continuity, they are strictly monotonic on $[a,b]$. Thus, g, h must be finite on $[a,b]$, for if $g(b) = \pm\infty$, then

$$\eta(x) = g(b) + h(x) = g(b) + h(y) = \eta(y)$$

for all x, y in (a,e). In the same way, $h(b) \neq \pm\infty$.

Four cases are possible.

<u>Case (i).</u> Both g, h are increasing, so that

$$\text{Dom } f \supseteq D_1 = [g(a)+h(a), g(b)+h(b)].$$

<u>Case (ii).</u> Both g, h are decreasing, so that

$$\text{Dom } f \supseteq D_2 = [g(b)+h(b), g(a)+h(a)].$$

<u>Case (iii).</u> g is increasing, h is decreasing, so that

$$\text{Dom } f \supseteq D_3 = [g(a)+h(b), g(b)+h(a)].$$

<u>Case (iv).</u> g is decreasing, h is increasing, so that

$$\text{Dom } f \supseteq D_4 = [g(b)+h(a), g(a)+h(b)].$$

Consider case (i). Now,

$$D_1 = [g(a)+h(a), g(b)+h(a)] \cup [g(b)+h(a), g(b)+h(b)].$$

Thus, for each x in $D_1$, either

(10) $\quad x = g(u)+h(a) \quad$ for some u in $[a,b]$,

or

(11) $\quad x = g(b)+h(v) \quad$ for some v in $[a,b]$.

By assumption,

(12) $\quad \bar{x} = T(\bar{x},\bar{x}) = f(g(\bar{x})+h(\bar{x}))$.

It follows from (8) that

(13) $\bar{x} = (f \circ \eta)\bar{x} = T(b,\bar{x}) = f(g(b)+h(\bar{x}))$.

Since $g(\bar{x})$, $h(\bar{x})$, $g(b)$ are finite and g is monotonic,

(14) $g(\bar{x})+h(\bar{x}) \neq g(b)+h(\bar{x})$.

Hence, by (10) and (11), either

(15) $g(\bar{x})+h(\bar{x}) = g(u)+h(a)$ for some u in I

or

(16) $g(\bar{x})+h(\bar{x}) = g(b)+h(v)$ for some $\nu$ in I.

By (14), $h(\bar{x}) \neq h(v)$; thus $v \neq \bar{x}$.

If (15) holds, then by (5)

(17) $\bar{x} = f(g(\bar{x})+h(\bar{x})) = f(g(u)+h(a)) = T(u,a) = a$

contradicting the assumption that $\bar{x}$ is in (a,b). If (16) holds, then

(18) $\bar{x} = f(g(\bar{x})+h(\bar{x})) = f(g(b)+h(v)) = T(b,v) = v \neq \bar{x}$.

Thus the theorem is proved for case (i).

Cases (ii), (iii), (iv) are handled similarly, upon noting that

$$D_2 = [g(b)+h(b), g(a)+h(b)] \cup [g(a)+h(b), g(a)+h(a)],$$

$$D_3 = [g(a)+h(b), g(a)+h(a)] \cup [g(a)+h(a), g(b)+h(a)],$$

$$D_4 = [g(b)+h(a), g(a)+h(a)] \cup [g(a)+h(a), g(a)+h(b)].$$

Thus the theorem is proved.

It should be noted that conditions (4) and (7) are equivalent to a very weak associativity requirement on T, for (4) and (7) imply

$$T(T(b,b),x) = T(b,x) = \phi(x) = \phi(\phi(x)) = T(b,T(b,x))$$

and

$$T(x,T(b,b)) = T(x,b) = \psi(x) = \psi(\psi(x)) = T(T(x,b),b).$$

Also, the theorem places no restriction whatsoever upon the behavoir of T in the interior of IxI; only boundary conditions are assumed.

## Bibliography

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