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METRIC SIMILARITIES IN THE LOGIC OF APPROXIMATION

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ABSTRACT

We describe restricted and extended versions of the "logic of approximation" which is meant to handle formally the problems of measurement error and of deduction under conditions of uncertainly. We apply the logic to the foundations of social and behavioral inquiry, axiomatizing in it an inexact similarity predicate which behaves like a metric approximation to identity. In the restricted version of the logic we formulate conditions for the imbeddability of similarity models in the real line, and in the extend version we relate arbitrary predicates of our formal language to the predicate of metric similarity.

§1. Similarity relations.

The notion of similarity is fundamental in the social and behavioral sciences. It is encountered in a wide variety of forms and contexts. In fact, similarity relations (a term we use to refer also to relations of indifference, proximity, co-variability, etc.) are, as Tversky [12] notes, ubiquitous in both theory and research (see also Gregson [3]).

Similarities are usually analyzed by means of metric models (see Shepard [11]) where distances represent dissimilarities. Hence, dissimilarities can be regarded as deviations from identity ("ideal similarity") and similarity as an approximation to identity. In other words, similarity is an identity relation in principle but a metric function in practice.

It is indeed the case that in many studies of similarity we have a (two-valued) relation in theory which becomes a (multi-valued) function in reality. For instance, in a typical experiment involving paired comparisons a subject may be asked a two-valued question - Are two given objects similar on not? - and yet, repetitions yield degrees of similarity (since some pairs will be judged similar more frequently than others) and these degrees usually lead to metric representations.

Similarly, in certain psychophysical experiments, where we try to find whether or not subjects can distinguish between various stimuli, it normally turns out that some pairs of stimuli are distinguishable more frequently than others, even by the same subject. Here, it is customary to reduce the multivalued phenomenon back to two-valuedness by stipulating that two stimuli are "indifferent" iff they are indistinguishable in more than half the times they are presented together (in which case we say that the "grap" between them is below the "differential threshold"). Yet, in connection to this type of indifference we also find studies (e.g., Roberts [10]) where gaps are represented by metric distances.

In this paper we present a logic for social inquiry where similarity has all the properties of identity in principle (i.e., in the formal language) but becomes a metric in practice (i.e., in an actual structure). We consider similarity, like identity in a formal logic for mathematics, an integral part of the logical framework itself, but we stress that the logic was not invented just to handle similarities. It is meant to treat formally the fundamental problem of measurement error and approximate deduction. It provides also plausible axiomatizations of various predicates in additon, and in connection, to the similarity predicate. This was shown in Katz [5], [6], [7], and will be partly repeated here.

We would like to note that the purpose of this paper is to bring together several ideas and theorems from the earlier works mentioned above. No new results will be proved, but some predicates and formulae will be given a new meaning, and the theory will be unified within the extended logical framework, under the interpretation of similarity as a general notion of metric approximation to identity. (Similar ideas, under a different interpretation of certain predicates, were applied to modal quantum logic in Katz [8]).

§ 2. Semantics.

Until we get to the definition of deductions in the extended logic of approximation we are working within a [0,1]-valued version of the familiar Lukasiewicz Logic, as can be found, e.g., in Lukasiewicz and Tarski [9]. This logic was proposed as a formal framework for the social sciences in Goguen [2] and for the physical sciences in Giles [1].

We interpret truth-values as degrees of error, and thus, like Giles, we let 0 (no error) stand for absolute truth and 1 (maximal error) for absolute falsehood. This effects also the semantic rules for compound formulae. We understand conjunction as maximizing the error and disjunction as minimizing it, and similarly for the quantifiers. The error in an implication statement is the difference between the truth-values of the consequent and the antecedent, and there is no error if the conseuqent is "truer than" the antecedent. Finally asserting the negation of a formula, φ, is equivalent to asserting that φ implies absolute falsehood.

To make all this precise, let L be a first-order language with a set V of variables, sets P_n of n-place predicate symbols for various n's in ω, the connectives $\neg, \wedge, \vee, \rightarrow$, the quantifiers $\forall$, $\exists$ and the auxiliary symbols of comma and parentheses. If $p \in P_n$ and $v_1, \ldots, v_n \in V$ then $p(v_1, \ldots, v_n)$ is an (atomic) formula of L. If ψ and θ are formula of L and $v \in V$ then $\neg\psi, \psi \wedge \theta$, $\psi \vee \theta$, $\psi \rightarrow \theta$, $\forall v \psi$, and $\exists v \psi$ are (compound) formulae of L. Parentheses may be added to compound formulae for ease of reading. The notions of free and bound variables are defined as usual.

A (multi-valued) structure $\underline{X}$ for L consists for a non-empty set X (the domain of $\underline{X}$) and for each $n \in \omega$ and each $p \in P_n$ a [0,1] -valued function $\underline{p}$ on X^n (the interpretation of p in $\underline{X}$). If φ is a formula of L, U is a finite subset of V containing all free variables of φ and $\bar{x} \in X^U$, then $\varphi\bar{x}$ denotes the truth-value of φ at $\bar{x}$. These values are defined recursively as follows:

(a) If $p \in P_n$, $v_1, \ldots, v_n \in V$ and $\underline{p}$ is the interpretation of p in $\underline{X}$ then

$$p(v_1, \ldots, v_n)\bar{x} = \underline{p}(\bar{x}(v_1), \ldots, \bar{x}(v_n)).$$

(b) If ψ and θ are formulae of L and for any real numbers ε and δ, here and throughout the paper, we write $\varepsilon \dot{-} \delta$ for $\max(0, \varepsilon - \delta)$ then

$$
\begin{aligned}
(\neg\psi)\bar{x} &= 1 - \psi\bar{x} \\
(\psi \wedge \theta)\bar{x} &= \max(\psi\bar{x}, \theta\bar{x}) \\
(\psi \vee \theta)\bar{x} &= \min(\psi\bar{x}, \theta\bar{x}) \\
(\psi \rightarrow \theta)\bar{x} &= \theta\bar{x} \dot{-} \psi\bar{x}.
\end{aligned}
$$

(c) If ψ is a formula of L with free variables $v_1, \ldots, v_n$ and for $i \leqslant n$ we let

$$\psi\bar{x}(i/x) = \psi(\bar{x}(v_1), \ldots, \bar{x}(v_{i-1}), x, \bar{x}(v_{1+1}), \ldots, \bar{x}(v_n))$$

then for each $i \leqslant n$

$$(\forall v_i \psi)\bar{x} = \sup_{x \in X} \psi\bar{x}(i/x)$$

$$(\exists v_i \psi)\bar{x} = \inf_{x \in X} \psi\bar{x}(i/x),$$

and if the variable v of L is not free in ψ then

$$(\forall v \psi)\bar{x} = (\exists v \psi)\bar{x} = \psi\bar{x}.$$

We now want to define a semantic notion of deduction in our logic (and consider those deductions governing the similarity predicate). By "deductions" we mean expressions of the form $\Gamma \vdash \Delta$ (read - "from Γ deduce Δ") where Γ and Δ are finite lists of formulae of L. We first restrict attention to the case of an empty Γ, where we write $\vdash \Delta$ and remain, more or less, within the traditional Lukasiewicz Logic. We then proceed to a radically new treatment of the general case (first considered in Katz [7], [8]).

§3. The restricted logical framework.

Let L be a language as above, $\underline{X}$ a structure for L with domain X, Δ a finite list of formulae of L and U the set of all variables of L appearing free in members of Δ. We say that the deduction $\vdash \Delta$ holds in $\underline{X}$ (and that $\underline{X}$ is a model of $\vdash \Delta$, or that $\underline{X}$ satisfies $\vdash \Delta$) if for every $\bar{x} \in X^U$

$$(v\Delta)\bar{x} = 0,$$

where $v\Delta$ is the disjunction of all members of Δ.

We now stipulate that the language L contains a distinguished binary predicate symbol, s, representing similarity. A structure $\underline{X}$ for L is called a similarity model if it is a model of the following three deductions (where u,v,w are variables of L and [RE], [SY], [TR] stand for reflexivity, symmetry, transitivity, respectively):

$$[RE] \qquad \vdash s(u,u)$$

$$[SY] \qquad \vdash s(u,v) \rightarrow s(v,u)$$

$$[TR] \qquad \vdash s(u,v) \rightarrow (s(v,w) \rightarrow s(u,w))$$

It is clear that if $\underline{s}$ is the interpretation of s in $\underline{X}$ then $\underline{X}$ is a similarity model iff for all x,y,z, in the domain X of $\underline{X}$ we have

$$[\underline{RE}] \qquad \underline{s}(x,x) = 0$$

$$[\underline{SY}] \qquad \underline{s}(x,y) = \underline{s}(y,x)$$

$$[\underline{TR}] \qquad \underline{s}(x,y) \geqslant \underline{s}(x,z) \dot{-} \underline{s}(y,z).$$

Thus, s is an equivalence relation in L but $\underline{s}$ is a pseudo-metric in $\underline{X}$ (recall that $\underline{s}$ ranges in [0,1]). And note that [TR] is weaker than

$$[TR]^* \qquad \vdash (s(u,v) \wedge s(v,w)) \rightarrow s(u,w).$$

whereas in two-valued logic the two deductions are equivalent. (In general, it is well known, and easy to check, that $\vdash \psi \rightarrow (\theta \rightarrow \chi)$ is weaker than $\vdash (\psi \wedge \theta) \rightarrow \chi$ in multi-valued logic, while the two are equivalent in the two-valued case). We cannot replace [TR] by the stronger [TR]* in our theory of similarity, because we do not wish to reduce $\underline{s}$ to an ultrametric on X (see Jardine, Jardine and Sibson [4]) by having for all $x,y,z \in X$

$$[TR]^* \qquad \max(\underline{s}(x,y), \underline{s}(y,z)) \geqslant \underline{s}(x,z).$$

To make s more like identity, rather than merely an equivalence, in L, we have to deal with substitutivity. But we leave this to the extended logic in the following section. Here we turn now to another interesting point - we show that it is possible to formulate in L conditions which make similarity unidimensional (or linear, i.e., imbeddable in the real line).

Let L,s,X̲,X,s̲ be as above and let u,v,w,t, be variables of L. The similarity model X̲ is called a linear similarity model if it satisfies the following two linearity axioms:

$$[LN]_1 \qquad \vdash (s(u,v) \to s(v,w)) \to s(w,u),$$
$$(s(w,u) \to s(u,v)) \to s(v,w),$$
$$(s(v,w) \to s(w,u)) \to s(u,v).$$

$$[LN]_2 \qquad \vdash \big(((s(v,w) \to s(w,u)) \to s(u,v)) \wedge ((s(t,w) \to s(w,u)) \to s(u,t))\big) \to$$
$$\big(((s(v,t) \to s(t,u)) \to s(u,v)) \vee ((s(t,v) \to s(v,u)) \to s(u,t))\big).$$

Clearly, X̲ is a linear similarity model iff in addition to [R̲E̲], [S̲Y̲], and [T̲R̲] we have for each $x,y,z,p \in X$:

$$[\underline{LN}]_1 \text{ either } \underline{s}(z.x) \leqslant \underline{s}(y,z) \dot{-} \underline{s}(x,y)$$
$$\text{or } \underline{s}(y,z) \leqslant \underline{s}(x,y) \dot{-} \underline{s}(z,x)$$
$$\text{or } \underline{s}(x,y) \leqslant \underline{s}(z,x) \dot{-} \underline{s}(y,z)$$

$$[\underline{LN}]_2 \quad \max(\underline{s}(x,y) \dot{-} (\underline{s}(z,x) \dot{-} \underline{s}(y,z)),\ \underline{s}(x,p) \dot{-} (\underline{s}(z,x) \dot{-} \underline{s}(p,z))) \geqslant$$
$$\min(\underline{s}(x,y) \dot{-} (\underline{s}(p,x) \dot{-} \underline{s}(y,p)),\ \underline{s}(x,p) \dot{-} (\underline{s}(y,x) \dot{-} \underline{s}(p,y))).$$

We can now state the imbedding theorem, which shows that linearity of s is really what we mean by it, namely that s̲ can be represented by absolute-value distances on the real line.

Theorem 1. The structure X̲ for L (which domain X and interpretation s̲ of s) is a linear similarity model iff there is a real-valued function f on X s.t. for all $x,y \in X$

$$\underline{s}(x,y) = |f(x) - f(y)|.$$

The proof of this theorem can be found in Katz [5], where we also show that the seemingly monstrous $[LN]_2$ is needed only when there are exactly four equivalence classes of X mod $\underline{s} = 0$ (in all other cases it follows from the remaining deductions). In the sufficiency part of the proof we obtain the functions f by fixing arbitrary x and y from two different equivalence classes of X (if there are fewer than two such classes the case is trivial) and setting for each $z \in X$:

$$f(z) = \begin{cases} \underline{s}(x,z) & \text{if } \underline{s}(y,z) \leqslant \max(\underline{s}(x,y), \underline{s}(x,z)) \\ -\underline{s}(x,z) & \text{if } \underline{s}(y,z) > \max(\underline{s}(x,y), \underline{s}(x,z)). \end{cases}$$

It can be shown that, in the presence of [$\underline{RE}$], [$\underline{SY}$], and [$\underline{TR}$], the linearity conditions $[\underline{LN}]_1$ and $[\underline{LN}]_2$ are equivalent, respectively, to

$[\underline{LN}]_1^*$ either $\underline{s}(x,z) = \underline{s}(x,y) + \underline{s}(y,z)$

or $\underline{s}(x,y) = \underline{s}(x,z) + \underline{s}(z,y)$

or $\underline{s}(y,z) = \underline{s}(y,x) + \underline{s}(x,z)$

$[\underline{LN}]_2^*$ if $\underline{s}(y,p) = \underline{s}(x,z) = \underline{s}(x,y) + \underline{s}(y,z)$

and $\underline{s}(y,z) = \underline{s}(x,p)$

and $\underline{s}(x,y) = \underline{s}(p,z)$

then $\underline{s}(x,p) = 0$ or $\underline{s}(x,y) = 0$.

We prove that the function f as defined above satisfies the claim of the representation theorem by checking for every p and z of X all their possible "locations" under $[\underline{LN}]_1^*$ w.r.t. the fixed x and y. There are quite a few cases to check and only in one of them we have to apply $[\underline{LN}]_2^*$ to obtain the required result.

This is the case where

$$\underline{s}(x,z) = \underline{s}(x,y) + \underline{s}(y,z)$$
$$\underline{s}(p,y) = \underline{s}(p,x) + \underline{s}(x,y)$$
$$\underline{s}(x,z) = \underline{s}(x,p) + \underline{s}(p,z)$$
$$\underline{s}(p,y) = \underline{s}(p,z) + \underline{s}(z,y),$$

which can be handled without $[\underline{LN}]_2^*$ if x,y,z,p, do not belong to four pairwise distinct equivalence classes, or if there is a $q \in X$ s.t. x,y,z,p,q, belong to five pairwise distinct equivalence classes.

§ 4. The extended logical framework.

In two-valued logic the deduction $\Gamma \vdash \Delta$ is said to hold in a structure $\underline{X}$ for L if whenever all members of Γ are true in $\underline{X}$ at least one member of Δ is also true in $\underline{X}$. This, of course, ignores the possibility of error and inexactness. An obvious way of extending this idea to our multi-valued logic is to say that $\Gamma \vdash \Delta$ holds in $\underline{X}$ if for every $\bar{x} \in X^U$ (with X being the domain of $\underline{X}$ and U the set of all free variables of members of $\Gamma \cup \Delta$)

$$(\wedge\Gamma)\bar{x} \geqslant (\vee\Delta)\bar{x}$$

where $\wedge\Delta$ is the conjunction of all members of Γ and $\vee\Delta$ the disjunction of all members of Δ.

This definition does take errors into account, but it commits the opposite "sin" of ignoring our ability in scientific research to reduce the error almost as much as we wish without completely eliminating it. Also, this is not really an extension of the one-sided deduction of the preceding section, since any structure $\underline{X}$ is a model of $\Gamma \vdash \Delta$ iff it is a model of $\vdash \wedge\Gamma \rightarrow \vee\Delta$.

We now introduce a proper extension of the one-sided deduction, which takes into account both the inevitable existence of error and the possibility of reducing the size of the error. It formalizes the notions of approximate truth and approximate deduction, giving a mathematical content to the assertion that "some members of Δ are very nearly true provided all members of Γ are true enough".

With notations as above we say that $\Gamma \vdash \Delta$ holds in $\underline{X}$ (and $\underline{X}$ is a model of $\Gamma \vdash \Delta$) if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $\bar{x} \in X^U$

$$(\wedge \Gamma)\bar{x} < \delta \Rightarrow (\vee \Delta)\bar{x} < \varepsilon .$$

(Here, and in the sequel, ε and δ denote real numbers and $\Rightarrow$ denotes implication in the meta-language).

Note that we now have a proper extension of all three types of deduction mentioned above (the two-valued, the "obvious" multi-valued and the one-sided multi-valued). With the new $\vdash$ any structure $\underline{X}$ for L is a model of $\Gamma \vdash \Delta$ whenever it is a model of $\vdash \wedge\ \Gamma \rightarrow \vee \Delta$, but not the other way round.

Returning to the similarity predicate, s, we axiomatize substitutivity using the extend deduction format. For any variable w of L, any formula φ of L with free variables $u_1, \ldots, u_n$ and any $i \leqslant n$ we write $\varphi(u_1, \ldots, u_n)(i/w)$ for the formula obtained from φ by substituting w for each free occurrence of u_i in φ. Then substitutivity is the deduction

$$[SU] \qquad s(u,v) \vdash \varphi(u_1, \ldots, u_n)(i/u) \rightarrow \varphi(u_1, \ldots, u_n)(i/v)$$

for any variables u and v of L and each $i \leqslant n$.

Inside a similarity model $\underline{X}$ with domain X substitutivity becomes a uniform continuity condition w.r.t. the interpretation $\underline{s}$ of s in $\underline{X}$. For $\underline{X}$ is a model of [SU] iff for each $i \leqslant n$ and each $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $x, y \in X$ and all $\bar{x} \in X^{\{u_1, \ldots, u_n\}}$:

$$[\underline{SU}] \qquad s(x,y)<\delta \Rightarrow |\varphi\bar{x}(i/x) - \varphi\bar{x}(i/y)| < \varepsilon.$$

(Here, $\varphi\bar{x}(i/x)$ and $\varphi\bar{x}(i/y)$ are to be read the way $\psi\bar{x}(i/x)$ was read in the semantic rules for quantified formulae, and the absolute value is obtained from [$\underline{SY}$]).

The interpretation [$\underline{SU}$] of [SU] is, in fact, stronger than uniform continuity; it is a condition of equi-continuity across coordinates. Yet, under this interpretation, as the following theorem shows, [SU] generalizes from atomic to arbitrary formulae of L, which is not the case under the still stronger interpretation [SU] would have had in the "obvious" multi-valued extension of $\vdash$.

<u>Theorem 2.</u> If [SU] holds in a similarity model, $\underline{X}$, for every atomic formula of L then it holds in $\underline{X}$ for every formula of L.

The proof is by induction on the structure of φ and the tricky steps are those involving the quantifiers. For instance, suppose φ is of the form $\forall v_j\psi$ where ψ is a formula of L with free variables $v_1,\ldots,v_n$ and $j \leqslant n$. Then the induction hypothesis is that for every $i \leqslant n$ and each $\varepsilon>0$ there is a $\delta>0$ s.t. for all $x,y,z\in X$ and all $\bar{x}\in X^{\{v_1,\ldots,v_n\}}$:

$$\underline{s}(x,y)<\delta \Rightarrow |\psi\bar{x}(i/x)(j/z) - \psi\bar{x}(i/y)(j/z)| < \varepsilon$$

and we want to show that for every $i \leqslant n$ and each $\varepsilon>0$ we can find a $\delta>0$ s.t. for all $x,y\in X$ and all $\bar{x}\in X^{\{v_1,\ldots,v_n\}}$:

$$\underline{s}(x,y)<\delta \Rightarrow |\sup_{z\in X}\psi\bar{x}(i/x)(j/z) - \sup_{z\in X}\psi\bar{x}(i/y)(j/z)| < \varepsilon.$$

The essentiality of the cross-coordinate equi-continuity in this part of the proof is obvious, and the details can be found in Katz [7].

From now on we shall use the term similarity model to refer only to those structures which in addition to [RE], [SY], and [TR] satisfy also [SU] for all predicates of L (and hence, by the theorem above, for all formulae of L). Thus, for a similarity model the predicate s has all the formal properties of identity, and the interpretation of s in the model is a pseudo-metric w.r.t. which all formulae of L are uniformly continuous.

The interpretation of substitutivity as continuity seems to be quite plausible in social and psychological theory. It implies that if two objects are very similar then assertions in L are about equally true for the two of them (and see the discussion of such an idea in, e,g., Tversky and Russo [13]). Assertions having the almost inverse property of being very nearly true for two objects only if the two objects are quite similar, will be studied in the following section.

§5. Unidimensional properties.

Unidimensionality of similarity was discussed at the end of Section 3 above. Here we deal with unidimensionality of other predicates (and formulae) of L; but it is another type of unidimensionality.

We say that the formula φ of L is unidimensional in the similarity model $\underline{X}$ if $\underline{X}$ satisfies the deduction

$$[\text{UD}] \qquad \varphi(u), \varphi(v) \vdash s(u,v).$$

Here φ is assumed to have only one free variable. (If this variable is u we write $\varphi(u)$ for φ; if it is v we write $\varphi(v)$ for φ). This restriction is inessential, and we impose it merely in order to simplify our notations. Everything we do in this section can be carried out for formulae with any number of free variables.

The idea behind [UD] is this. If the formula φ expresses a certain property of objects (e.g., "goodness") and if this property is multidimensional (e.g., if there are several different criteria for goodness) then two objects may very nearly satisfy φ and yet be quite dissimilar (they are "good" by different criteria). Therefore, if every two objects almost fully satisfying φ must be very similar to each other it makes sense to say that the property expressed by φ is unidimensional.

The deduction [UD] captures neatly the idea above. If X and $\underline{s}$ are, as before, the domain of the similarity model $\underline{X}$ and the interpretation of s in $\underline{X}$, then $\underline{X}$ is a model of [UD] iff for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. for all $x, y \in X$

$$[\underline{UD}] \qquad \max(\varphi(x), \varphi(y)) < \delta \Rightarrow \underline{s}(x,y) < \varepsilon.$$

So, any object satisfying a unidimensional formula up to a very small error is unique up to a close similarity. Hence, it is easy to see that such a formula can be fully true for at most one object of X (mod $\underline{s}=0$), and, as the following theorem shows, this object uniquely determines the formula, up to deduction.

<u>Theorem 3.</u> If the formulae ψ and θ of L are both unidimensional in the similarity model $\underline{X}$ and there is an x_o in the domain X of $\underline{X}$ s.t.

$$\psi(x_o) = \theta(x_o) = 0$$

then both $\psi \vdash \theta$ and $\theta \vdash \psi$ hold in $\underline{X}$.

To prove this theorem, given an $\varepsilon > 0$ use $[\underline{SU}]$ to find a real number $\delta' > 0$ s.t. for all $x \in X$

$$\underline{s}(x, x_o) < \delta' \Rightarrow |\psi(x) - \psi(x_o)| < \varepsilon$$

where $\underline{s}$ is the interpretation of the similarity predicate s in $\underline{X}$. Then use $[\underline{UD}]$ to find a $\delta > 0$ s.t. for all $x \in X$

$$\max(\theta(x),\theta(x_o))<\delta \Rightarrow \underline{s}(x,x_o)<\delta'.$$

So, for an arbitrary $\varepsilon>0$ we found a $\delta>0$ s.t. for all $x\in X$

$$\theta(x) = \max(\theta(x),\theta(x_o))< \delta \Rightarrow \psi(x)=|\psi(x)-\psi(x_o)|<\varepsilon.$$

Hence $\theta \vdash \psi$ holds in $\underline{X}$, and in the same way we can show that $\psi \vdash \theta$ holds in $\underline{X}$.

Since unidimensional formulae are determined by their zero points it is natural to ask when do they have such points. Thus, we seek conditions on a similarity model $\underline{X}$ guaranteeing that for certain unidimensional φ's there are objects in the domain X of $\underline{X}$ which make these φ's fully true in $\underline{X}$. Existence of such objects can be axiomatized in L by the deduction

$$[EX] \qquad \vdash \exists v\varphi(v)$$

which obviously holds in $\underline{X}$ iff

$$[\underline{EX}] \qquad \inf_{x\in X} \varphi(x) = 0.$$

For φ's satisfying [$\underline{EX}$] to say that they have zero points is simply to say that they actually attain their infimums. It turns out that a sufficient condition for this (for unidimensional φ's) is that $(X,\underline{s})$ be a complete metric space. This condition becomes also necessary if L has "names" for all the functions mentioned in the second part of the following theorem.

<u>Theorem 4.</u> Let $\underline{X}$ be a similarity model with domain X and interpretation $\underline{s}$ of similarity. If $(X,\underline{s})$ is a complete metric space then for every unidimensional formula φ of L for which[EX] holds in $\underline{X}$ there is an $x_o\in X$ s.t. $\varphi(x_o) = 0$. Conversely, if for every [0,1]-valued function φ on X which satisfies [$\underline{SU}$], [$\underline{UD}$], and [$\underline{EX}$] there is an $x_o\in X$ s.t. $\varphi(x_o) = 0$ then $(X,\underline{s})$ is a complete metric

space.

The proof can be found in Katz [8]. In the sufficiency part we show that $\varphi(x_o) = 0$ where x_o is the limit of a Cauchy sequence $\{x_n : 1 \leqslant n < \infty\}$ of elements of X satisfying for each n

$$\varphi(x_n) < 1/n.$$

For necessity we take any Cauchy sequence of elements of X, use it to define a function φ on X by

$$\varphi(x) = \lim_{n\to\infty} \underline{s}(x, x_n)$$

for every $x \epsilon X$, show that if satisfies the required conditions and deduce that it has a zero-point which is also the limit of the sequence.

We conclude the paper with the mathematically interesting result that every similarity model is elementarily extendable (in the model-theoretic sense) to one in which every "appropriate" φ has a zero point. If $\underline{X}$ is a similarity model with domain X we let the "completion" of $\underline{X}$ be the structure $\hat{\underline{X}}$ for L whose domain $\hat{X}$ is the metric completion of X (w.r.t. the interpretation $\underline{s}$ of similarity in $\underline{X}$) and whose interpretation of each m-place predicate symbol p of L (for each $m \epsilon \omega$) is given by

$$\hat{\underline{p}}(\bar{x}) = \lim_{n\to\infty} \underline{p}(\bar{x}_n)$$

for every $x \epsilon \hat{X}^m$. (Here $\underline{p}$ is the interpretation of p in $\underline{X}$, $\{\bar{x}_n : 1 \leqslant n < \infty\}$ is a sequence of elements of X^m converging to $\bar{x}$, and we consider X a subset of $\hat{X}$, identifying x with the equivalence class of (x,x,...) for every $x \epsilon X$).

<u>Theorem 5.</u> Let $\underline{X}$ be a similarity model (with domain X) and let $\hat{\underline{X}}$ be its completion (with domain $\hat{X}$). Then:

(i) $\hat{\underline{X}}$ is a similarity model w.r.t. to the extension, $\hat{\underline{s}}$, of the interpretation of similarity from $\underline{X}$ to $\hat{\underline{X}}$.

(ii) Every formula of L satisfying [$\underline{UD}$] and [$\underline{EX}$] in $\hat{\underline{X}}$ w.r.t. $\hat{\underline{s}}$ has a zero point in $\hat{\underline{X}}$.

(iii) For every formula φ of L (with m free variables for some $m \in \omega$) and every $\bar{x} \in \hat{X}^m$, if $\{\bar{x}_n : 1 \leqslant n < \infty\}$ is a sequence of elements of X^m converging to $\bar{x}$ then the value of φ at $\bar{x}$ is

$$\hat{\varphi}(\bar{x}) = \lim_{n \to \infty} \varphi(\bar{x}_n).$$

In particular if $\bar{x} \in X^m$ then $\hat{\varphi}(\bar{x}) = \varphi(\bar{x})$, so that $\hat{\underline{X}}$ is an elementary extension of $\underline{X}$.

The proof of all three parts is given in Katz [7]. Part (iii) is verified by induction on the structure of φ, like Theorem 2 above, and the steps involving the quantifiers are the hard ones again. For instance, if φ is of the form $\exists u \psi(u, v_1, \ldots, v_n)$ where $u, v_1, \ldots, v_n$ are the free variables of ψ, u not necessarily the first of them, we use the induction hypothesis and the continuity of ψ to reduce the problem to showing that

$$\lim_{n \to \infty} \inf_{x \in X} \psi(x, \bar{y}_n) = \inf_{x \in X} \lim_{n \to \infty} \psi(x, \bar{y}_n)$$

where $\{\bar{y}_n : 1 \leqslant n < \infty\}$ is a sequence of elemrnts of X^m converging to the element $\bar{y}$ of $\hat{X}^m$. This is shown, for every $\bar{y} \in \hat{X}^m$, by a series of approximations using again the uniform continuity of ψ in $\hat{X}$. Similar arguments are used in the case of the universal quantifier.

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