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MEASURES OF FUZZINESS AND OPERATIONS WITH FUZZY SETS

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ABSTRACT

We discuss the effects that the usual set theoretic and arithmetic operations with fuzzy sets and fuzzy numbers have with respect to the energies and entropies of the fuzzy sets connected and of the resulting fuzzy sets, and we compare also the entropies and energies of the results of different such operations.

1. Introduction.

The problem of the measurement of fuzziness has many facets. And, hence, there are different ways to approach it. We do not intend to survey here these developments, but mention only some of the papers by de Luca/Termini [3],[4], Knopfmacher [9], Sugeno [15], Trillas/Riera [16] and ourselves [1] together with the surveys of e.g. Dubois/Prade [6], Prade [14]. There are many different points to look at those measures. We will consider here only the entropy and energy measures of deLuca/Termini [4], sometimes in a specialized form, becauce of the connection with the problems of

- evaluation of the difference between a fuzzy set and a crisp set (entropies).
- evaluation of the difference between a fuzzy set and a crisp singleton (energies and energy type measures)

(cf. also our [2]). And we will discuss the effects that usual operations with fuzzy sets or fuzzy numbers have with respect to the entropies and energies of the fuzzy sets connected, and will compare also the entropies and energies of the results of different such operations.

The notation we use is the standard one. Sometimes we consider some integrals with respect to some measures, in these cases we suppose once and for all in this paper - to avoid cumbersome discussions of mathematical subtleties - that these integrals will exist; and if we use the integral sign without explicit mentioning of the area of integration, this is the whole universe of discourse. R_+ is used for the set of non-negative real numbers.

2. Energy type measures and arithmetic operations with fuzzy numbers.

Fuzzy numbers A are models of inexactly known real numbers a. Therefore, if we ask for the "fuzziness" of the fuzzy number A we like to know something about how different this fuzzy number A is from the real number a - or, in technical terms: we ask for a difference between the fuzzy set A and the crisp singleton of a.

Of course, entropy measures cannot be used to get this answer. To see this, we have only to remark that the crisp fuzzy numbers - this are the well known interval numbers of interval mathematics [13] - are able to describe very inexactly known real numbers, but every crisp fuzzy number has entropy-value zero for eve ry entropy measure, like any crisp singleton. Instead, any function g evaluating the fuzziness of fuzzy numbers should have the following properties for fuzzy numbers A,B:

(1) If A is a crisp singleton then $g(A) = 0$.

(2) If $A \subseteq B$ then $g(A) \leqslant g(B)$.

For any such function g the function e_g defined by

$$e_g(A) = g(A) - g(\emptyset)$$

is an energy measure [2]. Therefore, functions g with properties (1),(2) will be called energy type measures. As in [2] we consider also more specialized energy type measures defined as

(3) $$g\ A = F\left(\int f(\mu_A(x))dx \right)$$

with monotonically increasing functions $f: R_+ \to R_+$ and $F: R_+ \to R_+$ with properties

$$f(y) = 0 \quad \text{iff} \quad y = 0,$$

$$F(y) = 0 \quad \text{iff} \quad y = 0.$$

The integration is over the whole real line or over some interval fixed in advance.

If we consider, as usual, only such fuzzy numbers that are convex and normal, we have for energy type measures g of kind (3) the stronger property realized:

(4) $g(A) = 0$ iff A is a crisp singleton.

For such energy type measures we now will consider the problem to get information on the fuzziness of sum, product, difference, quotient and the fuzzy maximum $\widetilde{\max}$ (cf. [5]).

In each case we assume that our fuzzy numbers are normal fuzzy sets.

<u>Proposition 2.1.</u> For all fuzzy numbers A,B and each energy type measure g of kind (3) there hold true

(a) $g(-A) = g(A)$,
(b) $g(A + B) \geqslant \max\{g(A), g(B)\}$,
(c) $g(A - B) \geqslant \max\{g(A), g(B)\}$.

Proof. (a) For every x we have $\mu_{-A}(x) = \mu_A(-x)$ and hence $g(-A) = g(A)$ by (3).

(b) Assume $\mu_A(a) = 1$, then according to e.g. [11]

$$g(A+B) = F(\int f(\sup_y \min\{\mu_A(y), \mu_B(x-y)\})dx)$$
$$\geqslant F(\int f(\min\{\mu_A(a), \mu_B(x-a)\} dx)$$
$$= F(\int f(\mu_B(x-a))dx)$$
$$= g(B);$$

and in the same way also $g(A+B) \geqslant g(A)$. Hence (b).

(c) Now is an easy consequence of (b) and (a).

<u>Proposition 2.2.</u> Assume that there exists a function $k: R_+ \to R_+$ such that for all nonnegative reals x,y: $F(x \cdot y) = k(x) \cdot F(y)$. Assume also that $\mu_A(a) = 1$ and $\mu_B(b) = 1$, A and B given fuzzy numbers. Then there holds true for every energy type measure g of kind (3)

$$g(A \cdot B) \geqslant \max\{k(a) \cdot g(B), k(b) \cdot g(A)\}.$$

Proof. We have (cf. [11])

$$g(A\cdot B) = F(\int f(\sup_{\substack{y,z\\ x=y\cdot z}} \min\{\mu_A(y),\mu_B(z)\})dx)$$

$$= F(\int f(\sup_{y\neq 0} \min\{\mu_A(y),\mu_B(x/y)\})dx)$$

and therefore in case $a \neq 0$

$$\geqslant F(\int f(\mu_B(x/a))dx)$$

$$= F(a\int f(\mu_B(z))dz)$$

$$= k(a)\cdot g(B).$$

With $a = 0$ we get $k(a) = 0$ and hence $g(A\cdot B) \geqslant 0$ which obviuously holds true. By $A\cdot B = B\cdot A$ there follows also $g(A\cdot B) \geqslant k(b)\cdot g(A)$.

If we assume, as in this last proposition, the existence of a function k such that always

$$F(x\cdot y) = k(x)\cdot F(y)$$

holds true, this is the same as to assume that always

$$F(x) = c \cdot x^{\alpha}$$

with two parameters c,α. And then we have $k(x) = x^{\alpha}$. But we will use the F,k-notation instead of the parameters c,α.

Proposition 2.3: Let k,B be given as in proposition 2.2 and assume that there exists $d > 0$ such that

$$\mu_B(y) = 0 \text{ for every } y \text{ with } |y| < d.$$

Then there holds true

$$g(1/B) \leqslant k(d^{-2}).\ g(B).$$

If we assume furthermore that there exists $c > 0$ such that also

$$\mu_B(y) = 0 \text{ for every } y \text{ with } |y| > c,$$

then there holds true

$$g(1/B) \geqslant k(c^{-2}).g(B).$$

Proof. We have

$$\begin{aligned} g(1/B) &= F(\int f(\mu_{1/B}(x))dx) \\ &= F(\int_{-\infty}^{0} f(\mu_B(1/x))dx + \int_{0}^{\infty} f(\mu_B(1/x))dx) \\ &= F(\int f(\mu_B(y))y^{-2}dy) \end{aligned}$$

and by $\mu_B(y) = 0$ if $y^2 < d^2$, i.e. if $y^{-2} > d^{-2}$, we have $y^{-2} \leqslant d^{-2}$ for every y with $\mu_B(y) \neq 0$ and get

$$\begin{aligned} &\leqslant F(d^{-2}\int f(\mu_B(y))dy) \\ &= k(d^{-2}).\ g(B). \end{aligned}$$

If we have also $\mu_B(y) = o$ if $y^2 > c^2$, we have $y^{-2} \geqslant c^{-2}$ for every for every y with $\mu_B(y) \neq 0$ and get therefore also

$$\begin{aligned} g(1/B) &= F(\int f(\mu_B(y))y^{-2}dy) \\ &\geqslant F(c^{-2}\int f(\mu_B(y))dy) \\ &= k(c^{-2}).g(B). \end{aligned}$$

<u>Corollary 2.4.</u> With a,b,c,g and A,B as in the last propositions and with $b \neq 0$ we get also

$$g(A : B) \geqslant \max\{k(ac^{-2}).g(B), k(b^{-1}).g(A)\}.$$

Having thus found lower bounds for entropy type measures of the results of arithmetic operations for fuzzy numbers, it is natural to ask also for upper bounds. But, as long as no special assumptions are made concerning the growth of f, it is only possible to estimate for every x of the support $|A|$ of A: $f(\mu_A(x)) \leq f(1)$; in this way one gets, * any one of the arithmetic operations,

$$g(A*B) \leq F(f(1)\cdot lg(|A*B|))$$

with $lg(|A*B|) = \int_{|A*B|} dx$ which in the special case of an interval $|A*B|$ is the length of that interval. Remembering that fuzzy numbers are generalizations of the interval numbers of interval analysis, this kind of estimation goes by replacing each convex fuzzy number by the interval number with the same support, calculating with this interval numbers, and finally determining the measure of the result.

One can try to get sharper estimations by specializing f, F and also the kind of fuzzy numbers considered. E.g., one can suppose to have linear functions f,F, and can suppose also that A,B always are triangular fuzzy numbers (cf. [5]) which can be represented as triples of reals as in [8]:

$$A = (a',a,a''), \qquad B = (b',b,b'')$$

with $\mu_A(a) = 1$, $a' = \sup\{x < a \mid \mu_A(x) = 0\}$, $a'' = \inf\{x > a \mid \mu_A(x)=0\}$, and b,b',b'' defined in the same way.

Then one has for addition

$$g(A + B) \geq \frac{a - a'}{a+b - (a'+b')} (g(A) + g(B))$$

and of course again: $g(-A) = g(A)$. But already the product is problematic: generally it is not more triangular.

Furthermore, one can not find better upper bounds as in preceding cases, because the same arguments as before are correct now.

The fuzzy maximum $\widetilde{\max}$ we will discuss in the next section.

3. Energy measures and set theoric operations with fuzzy sets.

From our point of view it is now only an inessential difference if we consider energy type measures or emergy measures. Hence we will, different from the last section, consider mainly energy measures. Of these energy measures, a broad class of interesting ones can also be described by formula (3); the main differences between energy and energy type measures appears only for finite universes of discourse.

Now we consider fuzzy sets in general - not only fuzzy numbers - and ask for information on the fuzziness of unions, intersections, and cartesian products. For each one of these operations we consider four variants, connected with the following kinds of conjunction of fuzzy logic ($s.t\in[0,1]$):

$$s \wedge_1 t = \min\{s,t\},$$

$$s \wedge_2 t = s\cdot t,$$

$$s \wedge_3 t = \max\{0,\ s + t - 1\},$$

$$s \wedge_4 t = \begin{cases} s, & \text{if } t = 1 \\ t, & \text{if } s = 1 \\ 0, & \text{if } s,t < 1. \end{cases}$$

The variants $\wedge_1$, $\wedge_2$, $\wedge_3$ were known and used already in the earlier years of the development of fuzzy sets theory; $\wedge_4$ was in troduced by D. Dubois in 1979 (cf. [12]). Corresponding disjunctions v_i, $i = 1,\dots,4$, are defined via deMorgan laws with the usual negation $\neg s = 1 - s$ as:

$$s \, v_1 \, t = \max\{s,t\},$$

$$s \, v_2 \, t = s + t - s \cdot t,$$

$$s \, v_3 \, t = \min\{1, s + t\},$$

$$s \, v_4 \, t = \begin{cases} s, & \text{if } t = 0 \\ t, & \text{if } s = 0 \\ 1, & \text{if } s,t>0. \end{cases}$$

With these propositional operators we define - using set theoretic notation as in [7] - the unions $\cup_i$, intersections $\cap_i$, and cartesian products x_i by:

$$A \cap_i B = \{x \| \mu_A(x) \wedge_i \mu_B(x)\},$$

$$A \cup_i B = \{x \| \mu_A(x) v_i \mu_B(x)\},$$

$$A \, x_i \, B = \{(x,y) \| \mu_A(x) \wedge_i \mu_B(y)\}.$$

Using these operations now we can discuss the fuzzy maxium of fuzzy numbers,

<u>Proposition 3.1.</u> For all fuzzy numbers A,B and energy or energy type measures g these holds true

$$g(A \cap_1 B) \leqslant g(\widetilde{\max}(A,B)) \leqslant g(A \cup_1 B).$$

Proof. By property (2) of g it is enough to prove the inclusions

$$A \cap_1 B \subseteq \widetilde{\max}(A,B) \subseteq A \cup_1 B.$$

Put $D = \widetilde{\max}(A,B)$. Then there holds for every point x of the universe of discourse

$$\mu_D(x) = \sup_{\substack{y,z \\ x=\max\{y,z\}}} \min\{\mu_A(y),\ \mu_B(z)\}.$$

Consider now

$$a = \inf\{x \mid \mu_A(x) = 1\}, \qquad b = \inf\{x \mid \mu_B(x)=1\}.$$

Then we have

$$\sup_{y \leqslant x} \mu_A(y) = \begin{cases} \mu_A(x), & \text{if } x < a \\ 1, & \text{if } a \leqslant x \end{cases}$$

and in the same way

$$\sup_{z \leqslant x} \mu_B(z) = \begin{cases} \mu_B(x), & \text{if } x < b \\ 1, & \text{if } b \leqslant x. \end{cases}$$

With this we get

$$\mu_D(x) = \max\{\min\{\mu_A(x) \sup_{z \leqslant x} \mu_B(z)\},$$
$$\min \sup_{y \leqslant x} \mu_A(y),\ \mu_B(x)$$

and therefore

$$\mu_D(x) = \begin{cases} \min\{\mu_A(x),\ \mu_B(x)\}, & \text{if } x < a \\ \mu_B(x), & \text{if } a \leqslant x \leqslant b \\ \max\{\mu_A(x),\ \mu_B(x)\}, & \text{if } b < x \end{cases}$$

and for every x

$$\min\{\mu_A(x),\ \mu_B(x)\} \leqslant \mu_D(x) \leqslant \max\{\mu_A(x),\mu_B(x)\}.$$

Going back to the set theoretic operations we have as a first result

<u>Proposition 3.2.</u> For any energy measure g and fuzzy sets A, B we have for $1 \leqslant i,j \leqslant 4$

(a) $g(A \cap_i B) \leqslant g(A \cap_j B)$ for $i \geqslant j$,

(b) $g(A \cup_i B) \leqslant g(A \cup_j B)$ for $i \leqslant j$,

(c) $g(A \cap_1 B) \leqslant \min\{g(A), g(B)\}$,

(d) $g(A \cup_1 B) \geqslant \max\{g(A), g(B)\}$.

Proof. It is easy to see that there holds true [12]

$$A \cap_4 B \subseteq A \cap_3 B \subseteq A \cap_2 B \subseteq A \cap_1 B \subseteq A,B,$$

$$A,B \subseteq A \cup_1 B \subseteq A \cup_2 B \subseteq A \cup_3 B \subseteq A \cup_4 B.$$

Together with property (2) of g this gives the results.

Proposition 3.3: Let g be any energy measure, A,B fuzzy sets. Then there holds true

$$g(A \times_4 B) \leqslant g(A \times_3 B) \leqslant g(A \times_2 B) \leqslant g(A \times_1 B).$$

Proof. There holds true $A \times_i B \subseteq A \times_j B$ for all $1 \leqslant j \leqslant i \leqslant 4$.

For the next result we need some more notation. We consider now energy (type) measures of our specialized kind. But, to be more general as with fuzzy numbers we assume that for the fuzzy subsets A of X, B of Y there are given - e.g. totally finite measures ν_1, ν_2 on X,Y such that our energy (type) measures g are given by

$$g(C) = F(\int f(\mu_C(x,y))d\nu_1 \times \nu_2) \tag{5}$$

for every fuzzy subset C of the universe X x Y. The corresponding energy (type) measures on the class of fuzzy subsets of X resp. Y we denote by g_1 resp. g_2. We also assume that A,B are given in such a way that we can apply the Fubini theorem to $\mu_{A x_i B}$. Finally, for any classical subset Z of X we denote

$$\nu_1(Z) = \int_Z d\nu_1$$

and in the same way with Y, ν_2; and we use also

$$A^1 = \{x \mid \mu_A(x) = 1\}, \qquad B^1 = \{y \mid \mu_B(y) = 1\}.$$

Proposition 3.4. For every energy measure g of kind (5) with a linear function F there is

$$g(A\ x_i\ B) \leqslant \min\{\nu_1(|A|)\cdot g_2(B), g_1(A)\cdot \nu_2(|B|)\}$$

for i = 1,2,3 and for i = 4 even

$$g(A\ x_4\ B) = g_1(A)\cdot\nu_2(B^1) = \nu_1(A^1)\cdot g_2(B).$$

Proof. For i = 1,2,3 we use in every case $\mu_A(x) \leqslant 1$ and $\mu_B(y) \leqslant 1$ together with the monotonicity of f,F. Thus, we give the calculation only for i = 2:

$$\begin{aligned} g(A\ x_2\ B) &= F(\int_X [\int_Y f(\mu_A(x)\cdot\mu_B(y))d\nu_2]d\nu_1) \\ &\leqslant F(\int_{|A|} [\int_Y f(\mu_B(y))d\nu_2]d\nu_1) \\ &= F(\int_{|A|} d\nu_1 \cdot \int f(\mu_B(y))d\nu_2) \\ &= \nu_1(|A|)\cdot g_2(B). \end{aligned}$$

Of course, in the same way

$$g(A \times_2 B) \leqslant \nu_2(|B|)\cdot g_1(A)$$

and hence the result for i = 2. With i = 4 we get

$$\begin{aligned} g(A \times_4 B) &= F(\int_X[\int_Y f(\mu_A(x)\wedge_4\mu_B(y))d\nu_2]d\nu_1) \\ &= F(\int_X[\int_{B^1} f(\mu_A(x))d\nu_2]d\nu_1) \\ &= F(\int_X[f(\mu_A(x))\cdot\int_{B^1} d\nu_2]d\nu_1) \\ &= F(\nu_2(B^1)\cdot\int f(\mu_A(x))d\nu_1) \\ &= \nu_2(B^1)\cdot g_1(A) \end{aligned}$$

and again in the same way

$$g(A \times_4 B) = \nu_1(A^1)\cdot g_2(B).$$

The inspection of these calculations shows that one can get a bit a sharper estimation with i = 1:

$$g(A \times_1 B) \leqslant \min\{1,\ f(1)\}\cdot g_1(A)\cdot\nu_2(|B|)$$

and, of course, in every case

$$g(A \times_i B) = g(B \times_i A).$$

In case i = 2 we have also another interesting result.

<u>Proposition 3.5.</u> For every energy measure g of kind (5) with multiplicative functions f and F, i.e. with $f(x\cdot y) = f(x)\cdot f(y)$ and $F(x\cdot y) = F(x)\cdot F(y)$ for all x,y, there holds true

$$g(A \times_2 B) = g_1(A)\cdot g_2(B).$$

Proof. In this case we have

$$g(A\ x_2\ B) = F(\int_X f(\mu_A(x))[\int_Y f(\mu_B(y))d\nu_2]d\nu_1)$$

$$= g_1(A)\cdot g_2(B).$$

4. Entropy measures and set theoretic operations with fuzzy sets.

Entropy measures give information on the difference between a fuzzy set and a crisp set - i.e., they measure "fuzziness" as deviation of crispness (cf. [4] and also [2]). Therefore, we discuss entropy measures only in connection with operations for fuzzy sets, and not in connection with arithmetic operations for fuzzy numbers, The fundamental properties of any entropy measure h are for every fuzzy subset A,B of X:

(6) $h(A) = 0$ iff A is a crisp set;

(7) if $A \leqslant B$ then $h(A) \leqslant h(B)$

with $\leqslant$ the sharpness-ordering given as [4]

$$A \leqslant B \text{ iff } (\forall x \in X)(\mu_A(x) \leqslant \mu_B(x) \text{ if } \mu_B(x) \leqslant 1/2,$$

$$\text{and } \mu_A(x) \geqslant \mu_B(x) \text{ if } \mu_B(x) \geqslant 1/2).$$

Again, a broad class of interesting entropy measures h can be defined by the formula

(8) $h(A) = F(\int f(\mu_A(x)) d\nu)$

with ν any measure on the universe of discourse X, $F: R_+ \to R_+$ and

$f:[0,1] \to R_+$ such that F is increasing with: $F(y) = 0$ iff $y = 0$, and f increasing on $[0,1/2]$, decreasing on $[1/2,0]$ with $f(0)=f(1)=0$. Additionally, one can assume that always $f(x) = f(1 - x)$; this has as a consequence the symmetry of h with respect to complementation:

$$h(-A) = h(A).$$

As in the case of energy type measures, entropy measures of kind (8) do not always have property (6) - which corresponds to property (4) of energy type measures - but only the weaker one

(9) if A is a crisp set then $h(A) = 0$.

Of course, this is caused by the existence of nonempty subsets of the universe of discourse with ν-measure zero; if $|A|$ is such a set then obviously $h(A) = 0$ also.

But this problem is not essential here, because in any one of the cases we discuss entropy measures of kind (8) we can avoid to consider such "pathological" nonempty fuzzy sets.

Proposition 4.1. For every entropy measure h, any fuzzy sets A,B, and any $i = 1,2,3$ we have

(a) $h(A \cap_4 B) \leqslant h(A \cap_i B)$;

(b) $h(A \cup_4 B) \leqslant h(A \cup_i B)$;

(c) $h(A \times_4 B) \leqslant h(A \times_i B)$.

Proof. Put $D = A \cap_4 B$. In case $\mu_D(x) \neq 0$ now we have that $\mu_A(x) = 1$ or $\mu_B(x) = 1$. Assume $\mu_A(x) = 1$. Then $\mu_D(x) = \mu_B(x)$ and also $\mu_{A \times_i B}(x) = \mu_B(x)$. Hence we get: $A \cap_4 B \leqslant A \cap_i B$, i.e. (a). In the same way, $A \cup_4 B \leqslant A \cup_i B$, i.e. (b), and also $A \times_4 B \leqslant A \times_i B$, i.e. (c).

Unfortunately, this is the only positive result we can get for the unions, intersections, and cartesian products we consider.

First let us discuss unions and intersections. We assume that we consider fuzzy subsets of the real line, and e.g. only of the interval [0,10]; furthermore, if necessary, we use the entropy measure h_o defined as

$$h_o(A) = \int_0^{10} f_o(\mu_A(x))dx,$$

i.e., we choose F = id in (8), with f_o given by

$$f_o(x) = 1 - |2x - 1|.$$

Now it is easy to find for every $1 \leqslant i \leqslant 4$ fuzzy subsets A,B of [0,10] with the properties that

$$h_o(A \cap_i B) < h_o(A), \; h_o(B) < h_o(A \cup_i B).$$

One has only to choose for every i e.g. for $0 \leqslant x \leqslant 5$: $\mu_A(x) = 0.25$ and $\mu_B(x) = 0$, and for $5 < x \leqslant 10$: $\mu_A(x) = 0$ and $\mu_B(x) = 0.25$.

It is not more difficult to find also fuzzy subsets A,B of [0,10] such that for every i:

$$h_o(A \cup_i B) < h_o(A), \; h_o(B) < h_o(A \cap_i B).$$

Now we choose A,B in such a way that for $0 \leqslant x \leqslant 5$: $\mu_A(x) = 1$ and $\mu_B(x) = 0.5$, and for $5 < x \leqslant 10$: $\mu_A(x) = 0.5$ and $\mu_B(x) = 1$.

Finally, we can get also fuzzy subsets of [0,10] such that for every i there holds true

$$h_o(A) < h_o(A \cup_i B), \; h_o(A \cap_i B) < h_o(B).$$

In this case we can put for $0 \leqslant x \leqslant 5$: $\mu_A(x) = 0$, and for all $5 < x \leqslant 10$: $\mu_A(x) = 1$, and $\mu_B(x) = 0.5$ for every x.

Therefore, it seems quite impossible to get inequalities between $h(A)$, $h(A \cup_i B)$, and $h(A \cap_i B)$ which are useful and easy to work with. But, one can try to compare some $h(A \cap_i B)$ with some other $h(A \cup_j B)$, $i \neq j$. Again, unfortunately, one can get examples for any one of the relations

$$h_o(A \cap_i B) < h_o(A \cup_j B)$$

and

$$h_o(A \cap_i B) > h_o(A \cup_j B)$$

with $i \neq j$, $1 \leqslant i,j \leqslant 3$. To see this, consider A,B as in the three examples above together with the following three further cases: $\mu_A(x) = \mu_B(x) = 0.25$ for every x; $\mu_A(x) = \mu_B(x) = 0.5$ for every x; and $\mu_A(x) = \mu_B(x) = 0.75$ for every x.

Without giving further examples we only mention the additional fact that the situation is the same with the cartesian products.

Hence, we see that the "drastic operations" - this name was used by Mizumoto [10] for the cases i=4 - of D. Dubois are these ones which produce the "most crisp" results; and all the other unions, other intersections, other cartesian products are generally incomparable with respect to the "fuzziness" of their results.

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