NOTA BREVE

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QUASI-METRIZATION AND COMPLETION FOR PERVIN'S QUASI-UNIFORMITY

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ABSTRACT

R. Stoltenberg characterized in [2] those quasi-uniformities which are quasi-pseudometrizable, as well as those quasi-metric spaces which have a quasi-metric completion. In this paper we follow Stoltenberg's work by giving characterizations for quasi-metrizability and quasi-metric completion for a particular type of quasi-uniform spaces, the Pervin's quasi-uniform space.

Introduction.

In the following $(X,\mathcal{T})$ will be a topological space. $\mathcal{U}_p$ will be Pervin's quasi-uniformity associated to $(X,\mathcal{T})$, i.e., the one generated by the subbase $\mathcal{S}=\{S_G:G\in\mathcal{T}\}$, where $S_G=(G\times G)\cup(X\sim G\times X)$. $(X,\mathcal{U}_p)$ will denote Pervin's quasi-uniform space. The elements of a quasi-uniformity will be called bands; if U is a band U(x) will represent the set $\{y\in X:(x,y)\in U\}$.

We say that $(X,\mathcal{T})$ is quasi-pseudometrizable if there is a quasi-pseudometric d on X whose deduced topology, $\mathcal{T}_d$, is $\mathcal{T}$. In an

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analogous way, $(X,\mathcal{U})$ is quasi-pseudometrizable if there is a quasi-pseudometric d on X whose deduced quasi-uniformity, $\mathcal{U}_d$, is $\mathcal{U}$.

Proposition 1. $(X,\mathcal{U}_p)$ is a quasi-pseudometrizable space if and only if $\mathcal{T}$ is a countable family.

Proof. If $\mathcal{U}_p$ is quasi-metrizable then it has a countable base, $\{U_n\}_{n=1}^{\infty}$, see [2]. For each n, there is some finite number of open subsets $G_1^n, G_2^n, \dots, G_{k_n}^n$ such that $\cap\{S_{G_i^n} : 1 \leqslant i \leqslant k_n\} \subset U_n$. Let's consider the families $\mathcal{S}_n = \{G_i^n : 1 \leqslant i \leqslant k_n\}$, $\mathcal{S} = \{\mathcal{S}_n : n \geqslant 1\}$ and let $\mathcal{J}(\mathcal{S})$ be the family of finite intersections of members of $\mathcal{S}$. Obviously, $\mathcal{J}(\mathcal{S})$ is countable and it suffices to show that each proper open subset of X is a finite union of members of $\mathcal{J}(\mathcal{S})$.

Let G be a proper open subset of X, then there exists n such that $U_n \subset S_G$. For each $x \in G$, the set $I_x = \{i : 1 \leqslant i \leqslant k_n, x \in G_i^n\}$ is non-empty, otherwise $X = \cap\{S_{G_i^n}(x) : 1 \leqslant i \leqslant k\} \subset U_n(x) \subset S_G(x) = G$ which is a contradiction. Next we prove that $\cap\{G_i^n : i \in I_x\} \subset G$; if $y \notin G$ then $(x,y) \notin S_G$, thus $(x,y) \notin \cap\{S_{G_i^n} : 1 \leqslant i \leqslant k_n\}$ and therefore $(x,y) \notin \cap\{S_{G_i^n} : i \in I_x\}$ consequently, $y \notin \cap\{S_{G_i^n}(x) : i \in I_x\} = \cap\{G_i^n : i \in I_x\}$. Consider now the family of subsets of $\{1,2,\dots,k_n\}$ given by $I_G = \{I_x : x \in G\}$; this family is finite and $G = \cup\{\cap\{G_i^n : i \in I_x\} : x \in G\} = \cup\{\cap\{G_i^n : i \in I_x\} : I_x \in I_G\}$.

Conversely, suppose $\mathcal{T}$ is countable, then so is the family of finite intersections of its members, therefore $\mathcal{U}_p$ admits a countable base (see [1]) and thus by Theorem 1.6, page 228 of [2] $\mathcal{U}_p$ is quasi-pseudometrizable.

Consequences.

1.1. If T is countable, (X,T) is quasi-pseudometrizable.

1.2. There are quasi-uniform spaces which are not quasi-pseudometrizable.

1.3. Pervin's quasi-uniformity U_p is quasi-metrizable if and only if X is T_1 and T is countable.

1.4. If U_p is quasi-metrizable then X is countable.

Proof. The three first statements are direct consequence the former proposition.

Define a correspondence f from X to T such that $f(x)=X\sim\{x\}$, since X is T_1 f is a one-to-one function. Countability of T implies the same for X.

How we are going to study the quasi-metric completion for any Pervin quasi-uniform space. Before doing this we need a few definitions and results given by R. Stoltenberg in [2], we use the same references:

2.1. Definition. A net $(S_n, n\in D)$ in a quasi-uniform space (X,U) is U-Cauchy iff for each $U\in U$ there exists $n_o\in D$ such that $(S_n, S_m)\in U$ whenever $m\geqslant n_o, n\geqslant n_o, m\not\geqslant n$ and $m,n\in D$.

2.2. Definition. A quasi-uniform space (X,U) is U-complete iff each U-Cauchy net converges to a point in X relative to T_U.

2.3. Definition. A quasi-uniform space (X,U) has a U-completion iff (X,U) can be embedded in a U^*-complete quasi-uniform space (X^*,U^*) in such a way that X is dense in X^* relative to T_{U^*}.

5.3. Theorem. A quasi-metric space (X,d) has a quasi-metric d-completion if and only if all d-Cauchy sequences in (X,d,d')

which converge relative to $\mathcal{T}_d$, also converge relative to $\mathcal{T}_d$. (X,d,d') denotes the bi-quasi-metric space where d'(x,y)=d(y,x).

In order to achieve our own results we introduce a new concept: $\mathcal{U}_p^s$ will denote Pervin's symmetric quasi-uniformity associated to $\mathcal{U}_p$, i.e., the one whose subbase is $\mathcal{S}'=\{S_G':G\in\mathcal{T}\}$ and $S_G'=(G\times G)\cup(X\times X\sim G)$. The topology $\mathcal{T}_s$ induced by $\mathcal{U}_p^s$ in X is usually different from $\mathcal{T}$ and it will be called the symmetric topology of $\mathcal{T}$. Needless to say, $(X,\mathcal{T}_s)$ will be the symmetric topological space of $(X,\mathcal{T})$.

Proposition 2. Let $(X,\mathcal{T})$ be a topological space. The family of $\mathcal{T}$-closed subsets of X is a base for the symmetric topology $\mathcal{T}_s$.

Proof. Consider F as a non-empty closed subset of $(X,\mathcal{T})$; let G=X~F. For each x in F we know that $S_G'(x)$ is a $\mathcal{T}_s$-neighborhood of x; if $y\in S_G'(x)$ then $(x,y)\in(G\times G)\cup(X\times X\sim G)$ and therefore $y\in X\sim G$, thus $S_G'(x)\subset F$ and F is $\mathcal{T}_s$-open.

Now we show that every $\mathcal{T}_s$-open set is a union of $\mathcal{T}$-closed sets. Let G' be an open set of $\mathcal{T}_s$ with $G'\neq X$. If $x\in G'$ then by construction of $\mathcal{T}_s$ there is a finite family $\{G_i\}_{i=1}^n$ of $\mathcal{T}$-open sets such that $\bigcap_{i=1}^n S_{G_i}'(x)\subset G'$. Now, $S_{G_i}'(x)$ is either X, if $x\in G_i$, or $X\sim G_i$ if $x\notin G_i$. Then if we call $I_x=\{i:1\leqslant i\leqslant n,\ x\notin G_i\}$, we obtain $\bigcap_{i=1}^n S_{G_i}'(x)$ is X, if $I_x=\emptyset$, or $\bigcap_{i\in I_x}(X\sim G_i)$ if $I_x\neq\emptyset$; but $G'\neq X$, so $I_x\neq\emptyset$. Then $x\in\bigcap_{i=1}^n S_{G_i}'(x)=\bigcap_{i\in I_x}(X\sim G_i)\subset G'$. Obviously, $G'=\bigcup_{x\in G'}\left(\bigcap_{i\in I_x}(X\sim G_i)\right)$, i.e., G' is a union of $\mathcal{T}$-closed sets, since $\bigcap_{i\in I_x}(X\sim G_i)$ is $\mathcal{T}$-closed.

Consequences.

2.1. If X is a T_1 space, then $\mathcal{T}_s$ is the discrete topology.

2.2. If X is a regular space, then $\mathcal{T}_s$ is finer than $\mathcal{T}$.

Proof. The first consequence is quite obvious from last proposition.

Now, if G is $\mathcal{T}$-open, for each x in G there is $G_1 \in \mathcal{T}$ such that $x \in G_1 \subset \bar{G}_1 \subset G$. The set $G_2 = X \sim \bar{G}_1$ is $\mathcal{T}$-open and $x \in X \sim G_2 \subset G$. Since $S'_{G_2}(x) = X \sim G_2$, then $S'_{G_2}(x) \subset G$ and thus G is $\mathcal{T}_s$-open, for it is a $\mathcal{T}_s$-neighborhood of each of its points.

Theorem.

If a Pervin quasi-uniform space $(X, \mathcal{U}_p)$ is quasi-metrizable then it has a quasi-metric completion.

Proof. Let d be the quasi-metric in X which generates $\mathcal{U}_p$ and therefore induces $\mathcal{T}$. The symmetric quasi-metric d' induces $\mathcal{T}_s$, since d' induces $\mathcal{U}_p^s$. By consequences 1.3 and 2.1 $\mathcal{T}_s$ is the discrete topology.

Now, let $\{x_n\}_{n=1}^{\infty}$ be a d-Cauchy sequence d'-convergent to x. Since $\mathcal{T}_s$ is the discrete topology, for some positive integer n_o we have $x_n = x$ for all $n \geq n_o$; clearly $\{x_n\}_{n=1}^{\infty}$ converges to x relative to $\mathcal{T}$, i.e., $\{x_n\}_{n=1}^{\infty}$ is d-convergent. The conclusion follows from Theorem 5.3 of [2].

References.

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