NOTA BREVE

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A SEMANTICAL HIERARCHY FOR MODAL FORMULAS

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ABSTRACT

In this paper a semantical partition, relative to Kripke models, is introduced for sets of formulas.

Secondly, this partition is used to generate a semantical hierarchy for modal formulas. In particular some results are given for the propositional calculi T and S4.

1. Given any formal system S, if a Tarskian semantics is imposed on S, the set F_S of the well-formed formulas (wffs) of S can obviously be divided into three subsets: the set of the valid formulas (VA_S), the set of the contingent formulas (CO_S), the set of the contradictory formulas (CN_S). The introduction of the Kripke models allows a more subtle partition of the set F_S. We have two more subsets: the sets of the credible and dialectial formulas (as we shall say), respectively denoted by CR_S and DI_S.

The definitions of the above sets are the following (here and throughout this paper, our notation follows [1]). Let be any wff of S, then:

$\alpha \in VA_S$ iff for every structure $\langle W,R,V\rangle$ and any $w \in W$, $V(\alpha,w)=1$.

$\alpha \in CR_S$ iff for every structure $\langle W,R,V\rangle$, there exists at least a world $w \in W$ s.t. $V(\alpha,w) = 1$, but there exists at least a structure $\langle W',R',V'\rangle$ and a world $w' \in W'$ s.t. $V'(\alpha,w') = 0$.

$\alpha \in CO_S$ iff there exist at least a model $\langle W,R,V\rangle$ for and a structure $\langle W',R',V'\rangle$ s.t. $V'(\alpha,w') = 0$ for every $w' \in W'$.

$\alpha \in DI_S$ iff for every structure $\langle W,R,V\rangle$ there exists at least a world $w \in W$ s.t. $V(\alpha,w) = 0$, but there exist a structure $\langle W',R',V'\rangle$ and a world $w' \in W'$ s.t. $V'(\alpha,w') = 1$.

$\alpha \in CN_S$ iff for every structure $\langle W,R,V\rangle$ and any $w \in W$, $V(\alpha,w)=0$.

The terms 'credible' and 'dialectical' are used only for their intuitive appeal. A wff α (not valid) is credible if whenever in a structure it is denied in a world, there is another world in which it is asserted; and a wff β (not contradictory) is dialectical if whenever it is asserted in a world of a given structure, it is denied in another world of the same structure.

It is straightforward that the sets VA_S, CR_S, CO_S, DI_S and CN_S are pairwise disjoint, and that their union is the set F_S.

Also, for any reasonable systems S, certainly the sets VA_S, CO_S and CN_S are not empty.

Finally, it is clear that, if for a system S the sets CR_S and DI_S are empty, the Kripkian semantics for S can be reduced to a Tarskian semantics.

2. In what follows, we want to apply the previous definitions to some modal calculi. The interesting result we are going to prove is the construction of semantical hierarchies for sets of modal formulas. We believe that this can be a useful contribution towards a rigorous definition of the intuitive notions of modal

complexity and of truthlikeness of a modal formula and towards an analysis of their mutual relationships. In this short note, however, we limit ourselves to give some results about the propositional calculi T and S4.

Theorem 1. For the systems T and S4, there exists a credible formula.

Proof. Let α be the formula pvL∿p, where p is a propositional variable. Obviously $\alpha \in VA_T$, $\alpha \in VA_{S4}$. Now, let <W,R,V> be any T-(S4-) model and suppose that V(pvL∿p,w) = 0 for every w∈W. Then we have V(p,w) = 0 and V(L∿p,w) = 0, and there exists w'∈W s.t. V(p,w') = 1, a contradiction.

Theorem 2. For T and S4, there exists a dialectical formula.

Proof. Immediate (see Table 1, below).

Now, we give some tables for the operator L and for the connectives ∿ and ⊃, which are valid in the systems T and S4. These tables can be easily understood and proved. Of course, tables for the operator M and for the other connectives can be also obtained.

α ∈ VA → ∿α ∈ CN	α ∈ VA → Lα ∈ VA
α ∈ CR → ∿α ∈ DI	α ∈ CR → Lα ∈ CR or Lα ∈ CO
α ∈ CO → ∿α ∈ CO	α ∈ CO → Lα ∈ CO
α ∈ DI → ∿α ∈ CR	α ∈ DI → Lα ∈ DI or Lα ∈ CN
α ∈ CN → ∿α ∈ VA	α ∈ CN → Lα ∈ CN
Table 1	Table 2

α ⊃ β	VA	CR	CO	DI	CN
VA	VA	CR	CO	DI	CN
CR	VA	VA/CR	VA/CO	CO/DI	DI
CO	VA	VA/CR	VA/CR/CO	CR/CO	CO
DI	VA	VA/CR	VA/CR	VA/CR	CR
CN	VA	VA	VA	VA	VA

Table 3

All the sets VA, CR, CO, DI, CN, relative to the systems T, S4, are decidable. The procedure runs in a very similar way to that used as a test for validity in T and in S4 (see [1]).

3. Now, we give the following definition:

Definition 1. Given a wff $\alpha \in CR_S$, we say that α is (h,k)-credible (relative to S), if h is the least integer such that $M^h\alpha \in VA_S$ and k is the least integer such that $L^k\alpha \in CO_S$.

Theorem 3. For every wff $\alpha \in CR_T$, there exist two integers h, k, s.t. α is (h,k)-credible relative to T.

Proof. Use at the beginning a procedure similar to that given in [1] for T-validity. So, let w_0 be a certain world and suppose that $V(\alpha,w_0) = 1$.

In this way a finite set $\bar{W}$ of worlds is constructed and for each $w \in \bar{W}$, some subformulas of α are given truth-values by means of the rules for truth-value assignments explained in [1]. Since $\alpha \in CR_T$, a world $w' \in \bar{W}$ exists s.t. $V(\alpha,w') = 1$ and w_0R^nw', for a suitable $n \in N$. From this result one can easily infer that in any T-model $\langle W,R,V \rangle$, $V(M^n\alpha,w) = 1$ for every $w \in W$.

Now, starting from each $\bar{w} \in \bar{W}$, such that $V(\alpha,\bar{w}) = 1$, generate a sequence $\bar{w}_0 = \bar{w}, \bar{w}_1, \bar{w}_2, \ldots$, satisfying the following conditions:

(i) $\bar{w}_1 R \bar{w}_{i+1}$ $(i \geqslant 0)$;

(ii) for every $i \geqslant 0$, if in $\bar{w}_i$ there are prescriptions for worlds accessible from $\bar{w}_i$, construct the world $\bar{w}_{i+1}$ fulfilling such prescriptions. Otherwise let $\bar{w}_{i+1} = \bar{w}_0$, and the procedure stops.

Every such sequence is obviously finite and moreover there is a finite number of them (one for each $\bar{w} \in \bar{W}$ s.t. $V(\alpha,\bar{w}) = 1$).

If m denotes the length of the longest one, we have built in this way a model for the wff $M^m\sim\alpha$. Therefore, $L^m\alpha \in CO_T$. We have proved that α is (h,k)-credible relative to T, where h is the least n for which $M^n\alpha \in VA_T$ and k is the least m for which $L^m\alpha \in CO_T$.

Now, let us consider the wffs $\gamma_{m,n}$ given by

$$p \vee L^m M^n \sim p \qquad (m>0,\ n \geqslant 0).$$

Notice that the wff of Theorem 1 is precisely $\gamma_{1,0}$.

<u>Theorem 4.</u> $\forall\ m,n : \gamma_{m,n} \in CR_T$.

<u>Theorem 5.</u> $M^h\gamma_{m,n} \in CR_T \qquad (\forall\ h < m)$.

$M^h\gamma_{m,n} \in VA_T \qquad (\forall\ h \geqslant m)$.

<u>Theorem 6.</u> $L^k\gamma_{m,n} \in CR_T \qquad (\forall\ k \leqslant n)$.

$L^k\gamma_{m,n} \in CO_T \qquad (\forall\ k > n)$.

The proofs of Theorems 4,5,6 are obtained by means of a method similar to that adopted in [1] as a test for T-validity, already mentioned. Some minor changes must be introduce in order to test whether a given wff has at least a T-model.

As a consequence of Theorem 5 and 6, the wffs $\gamma_{m,n}$ are (m,n+1)-credible relative to the system T.

So far, we have shown that the set CR_T splits into a denumerable collection of nonvoid <u>cells</u> depending on two indices. Observe that the same result can be found for the set DI_T by a duality argument.

The following picture shows to what extent the formulas belonging to each cell are <u>close</u> to the sets VA_T and CO_T and justifies the informal meaning suggested above to such hierarchy.

T-credible wffs

. . . .

. . . .

T-valid wffs

1,4 2,3 3,2 4,1

1,3 2,2 3,2

1,2 2,1

1,1

T-contingent wffs

4. Now, consider the system S4. Table 3 for the operator L and its dual for the operator M, together with the S4-theses $L^k\beta \equiv L\beta$ and $M^k\beta \equiv M\beta$ $(k \geqslant 1)$, imply that for every wff $\beta \in CR_{S4}$, only the following cases are a priori admissible:

1) $L\beta \in CR_{S4}$ and $M\beta \in CR_{S4}$

2) $L\beta \in CO_{S4}$ and $M\beta \in CO_{S4}$

3) $L\beta \in CR_{S4}$ and $M\beta \in VA_{S4}$

4) $L\beta \in CO_{S4}$ and $M\beta \in VA_{S4}$.

But cases 1) and 2) are ruled out by the following Theorem 7, while cases 3) and 4) do actually occur, as shown in Theorem 8.

Theorem 7. If $\beta \in CR_{S4}$, then $M\beta \in VA_{S4}$.

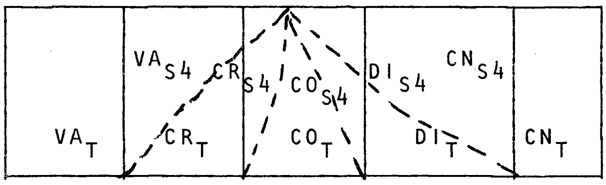
Theorem 8. There exist wffs $\alpha \in CR_{S4}$ such that $L\alpha \in CR_{S4}$ and there exist wffs $\beta \in CR_{S4}$ such that $L\beta \in CO_{S4}$.

The proof of Theorem 7 is immediate. In order to prove Theorem 8 we have only to exhibit two examples.

Example 1. Let α be a wff of the type $L^k\gamma_{m,n}$. If $\alpha \in CR_T \cap CR_{S4}$, and $L^{k+1}\gamma_{m,n} \in CO_T$, then $L\alpha \in CR_{S4}$.

Example 2. The wff $\gamma_{1,0} \in CR_{S4}$ and the wff $L\gamma_{1,0} \in CO_{S4}$.

Finally, the relation of inclusion which hold among the sets of wffs so far analyzed for the systems T and S4, can be visualized in the following picture



References.

[1] HUGHES, G.E. and CRESSWELL, M. J.- An Introduction to Modal Logic. Methuen, London, 1968.

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