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T-TOPOLOGIES ON A LATTICE ORDERED GROUP

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ABSTRACT

In this paper a characterization of the topologies on a ℓ-group arising from a CTRO (T-topologies) is given. We use it to find conditions under which the Redfield topology comes from a CTRO.

1. Introduction

Following Wirth [4], we shall say that T is a CTRO on an abelian ℓ-group G if T is a proper filter of G^+ such that T+T=T and $\wedge$T=0. From a CTRO T we can define a group and lattice topology $\mathcal{T}_T$ on G by taking the closed intervals [-t,t], where $t \in T$, as a neigbourhood base at zéro [2]. $\mathcal{T}_T$ is Hausdorff and not discrete. These properties happen to characterize the topologies coming from a CTRO which will be called T-topologies.

<u>Theorem 1.1.</u> $\mathcal{T}$ is a T-topology iff $\mathcal{T}$ is a non discrete Hausdorff group topology on G having a neigbourhood base at zero composed of closed intervals.

Proof: Let $\Sigma = \{[-a_\lambda, a_\lambda]\}_{\lambda \in \Lambda}$ be a neigbourhood base at zero for a non discrete Hausdorff group topology $\mathcal{T}$ on G. Define

$$T = \{x \in G \mid x \geq a_\lambda \text{ for some } \lambda \in \Lambda\}$$

Since $\mathcal{T}$ is not discrete it is clear that $0 \notin T$. T is a filter of G^+ because for every $\lambda, \mu \in \Lambda$ there exists $\gamma \in \Lambda$ such that

$$[-a_\lambda, a_\lambda] \cap [-a_\mu, a_\mu] = [-(a_\lambda \wedge a_\mu), a_\lambda \wedge a_\mu] \supset [-a_\gamma, a_\gamma].$$

Given $t \in T$ there exists $\lambda \in \Lambda$ such that $a_\lambda \leq t$ and so, being $\mathcal{T}$ a group topology, there is some $\mu \in \Lambda$ such that $a_\mu + a_\mu \leq t$. Then $a_\mu \leq t - a_\mu$ and $t = (t - a_\mu) + a_\mu \in T + T$. This proves that $T = T + T$. Finally, the fact of $\mathcal{T}$ being Hausdorff implies $\wedge T = 0$ and it is clear that $\mathcal{T} = \mathcal{T}_T$. ■

The goal of this paper is to give conditions under which theorem 1.1. can be applied to the Redfield topology [3]. The main result is stated in corollary 3.7. We confine attention to abelian groups, which we write additively.

An element $g \in G^+ \setminus \{0\}$ is <u>indecomposable</u> if $g_1 \vee g_2 = g$ and $g_1 \wedge g_2 = 0$ implies $g_1 = 0$ or $g_2 = 0$. Let a be the set of indecomposable elements of G. An element $g \in a$ is said to be <u>contractible</u> if there exists a sequence $\{g_n\}$ in a such that $g_1 = g$, $g_{n+1} + g_{n+1} \leq g_n$ and $g_{n+1}^\perp = g_n^\perp$ for each $n \in N$ ($\{g_n\}$ will be called a contractive sequence for g). Following Redfield's terminology in [3] let

$$D_1 = \{g \in a \mid g \text{ is contractible}\}$$

$$D_2 = \{g \in G^+ \setminus \{0\} \mid [0, g] = \{0, g\}\}$$

and $D = D_1 \cup D_2$. If $D \neq \emptyset$ a neigbourhood base at zero for the Redfield topology R is given by finite intersections of two kinds of sets:

1) $[-g, g] + g^\perp$ for $g \in D_1$

2) $g^\perp$ for $g \in D_2$

If $D=\emptyset$ R is defined as the indiscrete topology. In this way R is a group and lattice topology on any ℓ-group G.

If $D_1=\emptyset$ then R cannot have a neigbourhood base at zero composed of closed intervals and so R cannot arise from a CTRO. From now on we shall assume that $D_1\neq\emptyset$.

2. Basic neigbourhoods which are closed intervals.

A non empty set $I\subset G$ is a <u>solid ideal</u> of G if there exists a (lattice) ideal I^* of G such that $0\in I^*$ and $I=\{x\in G \mid |x|\in I^*\}$. The following lemmas have a straightforward proof in ℓ-group theory.

<u>Lemma 2.1.</u> Let I,J be solid ideals of G such that $I\cap J=\{0\}$. Then I+J is a solid ideal of G.

<u>Lemma 2.2.</u> Let I,J,K be solid ideals of G and $I\cap J=\{0\}$. Then

$$(I+J)\cap K = (I\cap K)+(J\cap K)$$

<u>Lemma 2.3.</u> Let $\{g_1,\dots,g_n\}\subset G^+\setminus\{0\}$ and, for each non empty $I\subset\{1,\dots,n\}$ let $A_I^n=[-\bigwedge_{i\in I} g_i, \bigwedge_{i\in I} g_i]\cap(\bigvee_{i\notin I} g_i)^{\perp}$. Then

$$A_I^n\cap A_J^n = \{0\}$$

for any couple of different non empty subsets I,J of $\{1,\dots,n\}$

<u>Proposition 2.4.</u> Let $\{g_1,\dots,g_n\}\subset G^+\setminus\{0\}$ and $N(0,g_i)=[-g_i,g_i]+g_i^{\perp}$ for $i=1,\dots,n$. Then

$$\bigcap_{i=1}^{n} N(0,g_i)=(\bigvee_{i=1}^{n} g_i)^{\perp}+\sum_{I\in P(n)} A_I^n$$

where $P(n)=\{I\subset\{1,\dots,n\} \mid I\neq\emptyset\}$ and A_I^n is defined as in lemma 2.3.

Proof. The assertion is clearly true for n=1 (it is assumed that $\bigvee_{i\in\emptyset} g_i=0$). If it is true for n-1 then

$$\bigcap_{i-1}^{n} N(0,g_i)=\{(\bigvee_{i=1}^{n-1} g_i)^{\perp}+\sum_{I\in P(n-1)} A_I^{n-1}\}\cap\{g_n^{\perp}+[-g_n,g_n]\}$$

All terms in this expression are solid ideals of G and we have

$$g_n^{\perp}\cap[-g_n,g_n]=\{0\}$$

$$(\bigvee_{i=1}^{n-1} g_i)^{\perp}\cap A_I^{n-1}=\{0\} \qquad \text{for each } I\in P(n-1)$$

and $A_I^{n-1}\cap A_J^{n-1}=\{0\}$ for each $I,J\in P(n-1)$ such that $I\neq J$, by lemma 2.3. So we can apply lemma 2.2 and obtain

$$\bigcap_{i=1}^{n} N(0,g_i)=(\bigvee_{i=1}^{n} g_i)^{\perp}+((\bigvee_{i=1}^{n-1} g_i)^{\perp}\cap[-g_n,g_n])+\sum_{I\in P(n-1)}(A_I^{n-1}\cap[-g_n,g_n])+\sum_{I\in P(n-1)}(A_I^{n-1}\cap g_n^{\perp})$$

But for each $I\in P(n-1)$,

$$A_I^{n-1}\cap[-g_n,g_n]=[-((\bigwedge_{i\in I} g_i)\wedge g_n),(\bigwedge_{i\in I} g_i)\wedge g_n]\cap(\bigvee_{i\notin I} g_i)^{\perp}=A^n_{I\cup\{n\}}$$

$$A_I^{n-1}\cap g_n^{\perp}=[-\bigwedge_{i\in I} g_i,\bigwedge_{i\in I} g_i]\cap((\bigvee_{i\notin I} g_i)\vee g_n)^{\perp}=A_I^n$$

and

$$(\bigvee_{i=1}^{n-1} g_i)^{\perp}\cap[-g_n,g_n]=A^n_{\{n\}}$$

Therefore

$$\bigcap_{i=1}^{n} N(0,g_i)=(\bigvee_{i=1}^{n} g_i)^{\perp}+\sum_{I\in P(n)} A_I^n$$

and, by induction, the assertion is true for any $n\in\mathbb{N}$. ■

The basic neigbourhoods at zero for R are sets U of the form

$$U(g_1\ldots g_n;\ g_{n+1}\ldots g_p)=(\bigcap_{i=1}^{n} N(0,g_i))\cap(\bigcap_{i=n+1}^{p} g_i^{\perp})$$

where $\{g_1,\ldots,g_n\} \subset D_1$ and $\{g_{n+1},\ldots,g_p\} \subset D_2$.

From proposition 2.4 and using lemmas 2.2 and 2.3 we can write

$$U(g_i\cdots g_n,g_{n+1}\cdots g_p)=(\bigvee_{i=1}^{p} g_i)^{\perp}+\sum_{I\in P(n)}(A_I^n\cap(\bigvee_{i=n+1}^{p} g_i)^{\perp})$$

In order to give conditions under which U is a closed interval we need some previous results.

<u>Lemma 2.5.</u> Let G be an archimedean ℓ-group, $f\in \mathcal{a}$ and $g\in D_2$.

If $f\geqslant g$ then $f=ng$ for some $n\in N$.

Proof: If $f=g$ the assertion is obvious. Assume $f>g$ and $f\neq ng$ for all $n\in N$. We shall prove, by induction, that $f>ng$ for all $n\in N$. If $f>ng$ we have $ng\geqslant ng\wedge(f-g)\geqslant(n-1)g$. But, if $ng\wedge(f-g)=(n-1)g$ it is $g\wedge(f-ng)=0$ and therefore $ng\wedge(f-ng)=0$ and $f=ng\vee(f-ng)$ in contradiction with the hypothesis $f\in\mathcal{a}$. Since $g\in D_2$, it must be $ng=ng\wedge(f-g)$ and so $f>(n+1)g$. G being archimedean, this implies $g=0$ which is a contradiction. ■

<u>Proposition 2.6.</u> Let G be an archimedean ℓ-group.

Then $D_1\subset D_2^{\perp}$.

Proof: Let $f\in D_1$, $g\in D_2$ and $\{f_n\}$ a contractive sequence for f. If $f\wedge g\neq 0$ then $f_n\wedge g\neq 0$ for each $n\in N$ and, since $g\in D_2$, it is $f_n\wedge g=g$. By lemma 2.5, for each $n\in N$ there must exist $k_n\in N$ such that $f_n=k_ng$. But $f_{n+1}+f_{n+1}\leqslant f_n$ for each $n\in N$ implies that $\{k_n\}$ is a strictly decreasing sequence of natural numbers what is not possible. So it must be $f\wedge g=0$ and $f\in D_2^{\perp}$. ■

<u>Corollary 2.7.</u> Let G be an archimedean ℓ-group, $A\subset D_1$ and $B\subset D_2$.

Then, $A^{\perp}\cap B^{\perp}=\{0\}$ iff $A^{\perp}=B^{\perp\perp}$

<u>Corollary 2.8.</u> Let G be an archimedean ℓ-group and $S\subset D$.

If $S^{\perp}=\{0\}$, then $D_2\subset S$.

Proof: If $g \in D_2$ there must exist $f \in S$ such that $f \wedge g > 0$. This implies $f \wedge g = g$. If $g \neq f$ it must be $f \in D_1$ and, by proposition 2.6, $g \wedge f = 0$. Hence, we have $g = f \in S$. ■

<u>Proposition 2.9.</u> Let G be an archimedean ℓ-group, $\{g_1, \ldots, g_n\} \subset D_1$ and $\{g_{n+1}, \ldots, g_p\} \subset D_2$.

Then

$$U(g_1 \ldots g_n;\ g_{n+1} \ldots g_p) = \Big(\bigvee_{i=1}^{p} g_i\Big)^{\perp} + \sum_{I \in P(n)} A_I^n$$

Proof: By proposition 2.6, $[-\bigwedge_{i \in I} g_i, \bigwedge_{i \in I} g_i] \cap \Big(\bigvee_{i=n+1}^{p} g_i\Big)^{\perp} = [-\bigwedge_{i \in I} g_i, \bigwedge_{i \in I} g_i]$.

Hence, for each $I \in P(n)$, $A_I^n \cap \Big(\bigvee_{i=n+1}^{p} g_i\Big)^{\perp} = A_I^n$. ■

<u>Proposition 2.10.</u> Let G be an archimedean ℓ-group, $\{g_1, \ldots, g_n\} \in D_1$ and $\{g_{n+1}, \ldots, g_p\} \subset D_2$.

Then

$U = U(g_1 \ldots g_n; g_{n+1} \ldots g_p)$ has an upper bound iff $\Big(\bigvee_{i=1}^{p} g_i\Big)^{\perp} = \{0\}$.

Proof: By proposition 2.9 it is clear that $\Big(\bigvee_{i=1}^{p} g_i\Big)^{\perp} \subset U$ and so $\Big(\bigvee_{i=1}^{p} g_i\Big)^{\perp} = \{0\}$ if U has an upper bound. On the other hand, if $\Big(\bigvee_{i=1}^{p} g_i\Big)^{\perp} = \{0\}$ we have $U = \sum_{I \in P(n)} A_I^n$. $g = \bigvee_{i=1}^{n} g_i$ is an upper bound for each A_I^n and, by lemma 2.3, g is an upper bound for U. ■

<u>Corollary 2.11.</u> Let G be an archimedean ℓ-group, $\{g_1, \ldots, g_n\} \subset D_1$ and $\{g_{n+1}, \ldots, g_p\} \subset D_2$.

If $U(g_1 \ldots g_n; g_{n+1} \ldots g_p)$ has an upper bound, then $\{g_{n+1}, \ldots, g_p\} = D_2$.

Proof: Proposition 2.10 and corollary 2.8. ■

If D_2 is finite we shall denote $U(g_1, \ldots, g_n; D_2)$ by $U(g_1 \ldots g_n)$.

Corollary 2.12. Let G be an archimedean ℓ-group, $\{g_1,\dots,g_n\}\subset D_1$ and D_2 finite.

Then $U(g_1,\dots,g_n)$ has an upper bound iff $(\bigvee_{i=1}^{n} g_i)^{\perp\perp} = D_2^{\perp}$.

Proof: By corollary 2.7 $(\bigvee_{i=1}^{n} g_i)^{\perp\perp} = D_2^{\perp}$ iff $(\bigvee_{i=1}^{n} g_i)^{\perp} \cap D_2^{\perp} = \{0\}$ and we can apply proposition 2.10. ■

Corollary 2.13. Let G be an archimedean ℓ-group, $\{g_1,\dots,g_n\}$ an ortogonal subset of D_1 and D_2 finite.

Then $U(g_1,\dots,g_n)$ is a closed interval iff $(\bigvee_{i=1}^{n} g_i)^{\perp\perp} = D_2^{\perp}$.

Proof: In this case $A_I^n=[-g_i,g_i]$ if $I=\{i\}$ for some $i\in\{1,\dots,n\}$ and $A_I^n=\{0\}$ otherwise. Hence,

$$\sum_{I\in P(n)} A_I^n = \sum_{i=1}^{n}[-g_i,g_i] = [-\bigvee_{i=1}^{n} g_i, \bigvee_{i=1}^{n} g_i]$$

If $(\bigvee_{i=1}^{n} g_i)^{\perp\perp} = D_2^{\perp}$, then $(\bigvee_{i=1}^{n} g_i)^{\perp} \cap D_2^{\perp} = \{0\}$ and, by proposition 2.9, $U(g_1\dots g_n)=[-\bigvee_{i=1}^{n} g_i, \bigvee_{i=1}^{n} g_i]$. ■

We shall see now that, under suitable conditions, the basic neigbourhoods $U(g_1,\dots,g_n)$ such that $(\bigvee_{i=1}^{n} g_i)^{\perp\perp} = D_2^{\perp}$ are the only basic neigbourhoods that are closed intervals.

Theorem 2.14. Let G be an archimedean ℓ-group such that $a\cup\{0\}$ is an inf-semilattice, $\{g_1\dots g_n\}\subset D_1$ and D_2 finite.

Then, $U(g_1,\dots,g_n)$ is a closed interval iff there exists an ortogonal subset $\{h_1,\dots,h_r\}\subset D_1$ such that $U(g_1,\dots,g_n)=U(h_1,\dots,h_r)$ and $(\bigvee_{i=1}^{r} h_i)^{\perp\perp} = D_2^{\perp}$.

For the proof of theorem 2.14, we need the following two lemmas:

Lemma 2.15. For any finite subset $\{g_1,\dots,g_n\}\subset G^{\perp}\setminus\{0\}$.

$$\bigcap_{I\in P(n)} (A_I^n)^{\perp} = (\bigvee_{i=1}^{n} g_i)^{\perp}$$

Proof: Let $x\in (A_I^n)^{\perp}$ for all $I\in P(n)$. If $|x|\wedge g_i\neq 0$ for some $i\in\{1,\dots,n\}$ then it is $|x|\wedge g_i\notin A_{\{i\}}^n$. But $|x|\wedge g_i\in[-g_i,g_i]$ and so, $|x|\wedge g_i\notin g_j^{\perp}$ for some $j\in\{1,\dots,n\}$, $j\neq i$. Now we have $|x|\wedge g_i\wedge g_j\neq 0$ and so, $|x|\wedge g_i\wedge g_j\notin A_{\{i,j\}}^n$. By repeating the reasoning above a finite number of times we obtain

$$|x|\wedge(\bigwedge_{i=1}^{n} g_i)\neq 0$$

what is a contradiction because $\bigwedge_{i=1}^{n} g_i\in A_{\{1,\dots,n\}}^n$. Hence,

$\bigcap_{I\in P(n)} (A_I^n)^{\perp}\subset(\bigvee_{i=1}^{n} g_i)^{\perp}$.

The other inclusion is clear. ∎

Lemma 2.16. If $\mathcal{a}\cup\{0\}$ is an inf-semilattice, then $D_1\cup\{0\}$ is an inf-semilattice.

Proof: Let $f,g\in D_1$ such that $f\wedge g>0$ and $\{f_n\}$ and $\{g_n\}$ contractive sequences for f and g resoectively. Then, $f_n\wedge g_n>0$ for all $n\in N$ and so, $f_n\wedge g_n\in\mathcal{a}$ for all $n\in N$. It is not difficult to prove that $\{f_n\wedge g_n\}$ is a contractive sequence for $f\wedge g$. Hence $f\wedge g\in D_1$. ∎

Proof of theorem 2.14: Assume $U=U(g_1,\dots,g_n)=[-u,u]$. By corollary 2.12, $(\bigvee_{i=1}^{n} g_i)^{\perp}=D_2^{\perp\perp}$ and by proposition 2.9 and corollary 2.7 it is $U=\sum_{I\in P(n)} A_I^n$. Being $u\in U$ there must exist, for all $I\in P(n)$, elements $u_I\in A_I^n, u_I\geq 0$, such that $u=\sum_{I\in P(n)} u_I$. By lemma 2.3 we have $u_I\wedge u_J=0$ if $I\neq J$. It is not difficult to prove that $A_I^n=[-u_I,u_I]$ for all $I\in P(n)$.

We shall see now that, given $I\in P(n)$, it is $u_I=0$ or $u_I=\bigwedge_{i\in I} g_i$. Since $u_I\in A_I^n$ we have $u_I\leq\bigwedge_{i\in I} g_i$. Let $x\geq 0$ be such that $x\leq\bigwedge_{i\in I} g_i-u_I$

and $x \leqslant u_I$. The last condition implies $x \in A_I^n$ and so, $mx \in (\bigvee_{i \notin I} g_i)^{\perp}$ for each $m \in N$ and $x \leqslant \bigwedge_{i \in I} g_i$. If $mx \leqslant \bigwedge_{i \in I} g_i$ then $mx \in A_I^n$ and it is $mx \leqslant u_I$. Therefore, $(m+1)x \leqslant u_I + \bigwedge_{i \in I} g_i - u_I = \bigwedge_{i \in I} g_i$. Hence, by induction, $mx \leqslant \bigwedge_{i \in I} g_i$ and we have $mx \leqslant u_I$ for all $m \in N$. This implies $x=0$ and so,

$$u_I \wedge (\bigwedge_{i \in I} g_i - u_I) = 0$$

$$u_I \vee (\bigwedge_{i \in I} g_i - u_I) = \bigwedge_{i \in I} g_i$$

By lemma 2.16, $\bigwedge_{i \in I} g_i \in D_1 \cup \{0\}$. Thus, $u_I = 0$ or $u_I = \bigwedge_{i \in I} g_i$.

Let $P'(n) = \{I \in P(n) \mid u_I \neq 0\}$. Then $\{u_I\}_{I \in P'(n)}$ is a finite ortogonal subset of D_1 such that $U(g_1, \ldots, g_n) = U(\{u_I\}_{I \in P'(n)})$. Furthermore, using lemma 2.15 we can write

$$(\bigvee_{I \in P'(n)} u_I)^{\perp} = \bigcap_{I \in P'(n)} u_I^{\perp} = \bigcap_{I \in P'(n)} [-u_I, u_I] = \bigcap_{I \in P(n)} [-u_I, u_I] = \bigcap_{I \in P(n)} (A_I^n)^{\perp} = (\bigvee_{i=1}^{n} g_i)^{\perp} = D_2^{\perp\perp}$$

what ends the proof. ■

3. Redfield topology and T-topologies.

Recall that an ortogonal set $S \subset G$ is maximal iff $S^{\perp} = \{0\}$. A maximal ortogonal set composed of elements of D will be called a <u>fundamental set</u>.

G es said to be <u>finitely founded</u> if each finite ortogonal subset of D_1 is contained in some finite fundamental set. By corollary 2.8 any fundamental set S contains D_2 and, by corollary 2.7, $(S \cap D_1)^{\perp} = D_2^{\perp\perp}$. As a consequence if G is finitely founded then D_2 is finite.

<u>Theorem 3.1.</u> Let G be an archimedean ℓ-group such that $a \cup \{0\}$ is an inf-semilattice.

Then, the basic neigbourhoods of zero that are closed intervals form a neigbourhood base at zero for R iff G is finitely founded.

To prove theorem 3.1 we need some previous results.

Lemma 3.2. If $g,f\in D_1$ and $g\wedge f>0$, then $g\vee f\in D_1$.

Proof: If $g\vee f\notin \mathfrak{a}$ there exists $h_1, h_2\in G^+\setminus\{0\}$ such that

$h_1 \vee h_2 = g \vee f$

$h_1 \wedge h_2 = 0$.

Then $g = (g\wedge h_1) \vee (g\wedge h_2)$ and $f = (f\wedge h_1) \vee (f\wedge h_2)$

$0 = (g\wedge h_1) \wedge (g\wedge h_2)$ $0 = (f\wedge h_1) \wedge (f\wedge h_2)$

But $g,f\in\mathfrak{a}$ which implies $g\wedge h_1 = 0$ or $g\wedge h_2 = 0$ and $f\wedge h_1 = 0$ or $f\wedge h_2 = 0$. Each of the four resultant possibilities leads to a contradiction. Hence, $g \vee f\in\mathfrak{a}$.

Let $\{g_n\},\{f_n\}$ be contractive sequences for g and f respectively and consider $\{g_n \vee f_n\}$. It is clear that $g_n\wedge f_n>0$ and so $g_n \vee f_n\in\mathfrak{a}$ for all $n\in N$. To prove $(g_{n+1}\vee f_{n+1})+(g_{n+1}\vee f_{n+1})\leqslant g_n\vee f_n$ represent G as a subdirect product of totally ordered groups $\{G^\lambda\}_{\lambda\in\Lambda}$. Then, given $\lambda\in\Lambda$, it is $g^\lambda_{n+1}\vee f^\lambda_{n+1} = f^\lambda_{n+1}$ or $g^\lambda_{n+1} \vee f^\lambda_{n+1} = g^\lambda_{n+1}$. In each case we obtain $(g^\lambda_{n+1}\vee f^\lambda_{n+1}) + (g^\lambda_{n+1}\vee f^\lambda_{n+1}) \leqslant g^\lambda_n\vee f^\lambda_n$. It is now easy to see that $\{g_n\vee f_n\}$ is a contractive sequence for $g\vee f$. Hence $g\vee f\in D_1$. ■

Lemma 3.3. Let G be a finitely founded ℓ-group.

Then, for each non empty finite set $F\subset D_1$ there exists a finite ortogonal set $S\subset D_1\cap F^\perp$ such that $(S\cup F)^\perp = D_2^{\perp\perp}$.

Proof: Let $F=\{f_1,\dots,f_n\}\subset D_1$ and define a finite ortogonal subset of D_1 in the following way: If $f_1\wedge f_i=0$ for all $1<i\leqslant n$ let $h_1=f_1$. Otherwise, there exists $f_i (1<i\leqslant n)$ such that $f_1\wedge f_i\neq 0$ and,

by lemma 3.2, $f_1 \vee f_i \in D_1$. If this element is ortogonal to any other element of F different from f_1 or f_i let $h_1 = f_1 \vee f_i$. Otherwise, repeat the procedure. By a finite number of steps we will have selected a subset $F_1 \subset F$ such that $h_1 = \vee F_1$ and $h_1 \in (F \setminus F_1)^{\perp}$. If $F_1 \neq F$ we pick an element in $F \setminus F_1$ and repeat the procedure. In this way we obtain a partition $F_1, \ldots, F_p$ of F and $H = \{h_1, \ldots, h_p\}$ is a finite ortogonal subset of D_1 such that $F^{\perp} = H^{\perp}$. Hence, there exists a finite fundamental set T which contains H. Then $S = (T \cap D_1) \setminus H$ satisfies $S \subset F^{\perp}$ and $(S \cup F)^{\perp} = D_2^{\perp\perp}$. ■

<u>Lemma 3.4.</u> Let G be a finitely founded ℓ-group such that $\mathfrak{a} \cup \{0\}$ is inf-semilattice.

Then, if $F = \{g_1, \ldots, g_n\} \subset D_1$ there exists a finite ortogonal set $H \subset D_1$ such that $F^{\perp} = H^{\perp}$ and $H \subset \sum_{I \in P(n)} A_I^n$.

Proof: Let $I = \{I \in P(n) \mid \bigwedge_{i \in I} g_i \in A_I^n\}$ and for $I \in I$ define $h_I = \bigwedge_{i \in I} g_i$. If $I \notin I$, $F_I = \{g_j \in F \mid g_j \notin (\bigwedge_{i \in I} g_i)^{\perp}\}$ is a non empty finite subset of D_1 and, by lemma 3.3, there exists a finite ortogonal set $S_I \subset D_1 \cap F_I^{\perp}$ such that $(S_I \cup F_I)^{\perp} = D_2^{\perp\perp}$. If $S_I \neq \emptyset$ define, for each $g_{I_k} \in S_I$, $h_{I_k} = g_{I_k} \wedge (\bigwedge_{i \in I} g_i)$. It is clear that $h_{I_k} \in A_I^n$ for all $g_{I_k} \in S_I$.

Let $H = \{h_I \mid I \in I,\ h_I \neq 0\} \cup \{h_{I_k} \mid I \notin I,\ h_{I_k} \neq 0\}$. H is a finite ortogonal subset of $\sum_{I \in P(n)} A_I$ which is contained in D_1 by lemma 2.16. It is not difficult to prove that $H^{\perp} = F^{\perp}$. ■

<u>Proof of theorem 3.1.</u> a) Assume that the basic neigbourhoods of zero that are closed intervals form a neigbourhood base at zero for R. By corollary 2.11 D_2 is, then, finite. Let $H = \{h_1, \ldots, h_p\}$ be an ortogonal subset of D_1. Then, by theorem 2.14, there exists an ortogonal subset $F = \{g_1, \ldots, g_n\}$ of D_1 satisfying $F^{\perp} = D_2^{\perp\perp}$ such that

$$U(g_1, \ldots, g_n) = [-\bigvee_{i=1}^{n} g_i, \bigvee_{i=1}^{n} g_i] \subset U(h_1, \ldots, h_p) = [-\bigvee_{j=1}^{p} h_j, \bigvee_{j=1}^{p} h_j] + (H \cup D_2)^{\perp}$$

Let $S=H\cup(F\cap(H\cup\mathcal{D}_2)^{\perp})\cup\mathcal{D}_2$. It is clear that S is a finite ortogonal subset of $\mathcal{D}$.

Since $F\subset\mathcal{a}$, for each $g_i\in F$, if $g_i\notin(H\cup\mathcal{D}_2)^{\perp}$ there exists a unique $h_j\in H$ such that $g_i\leq h_j$. Therefore

$$(H\cup(F\cap(H\cup\mathcal{D}_2)^{\perp})^{\perp}\subset F^{\perp}$$

and so, $S^{\perp}\subset F^{\perp}\cap\mathcal{D}_2^{\perp}$ what implies, by corollary 2.7, $S^{\perp}=\{0\}$. Hence, S is a fundamental set than contains H.

b) Suppose G finitely founded and let $F=\{g_1,\dots,g_n\}\subset\mathcal{D}_1$ and $\{g_{n+1},\dots,g_m\}\subset\mathcal{D}_2$. By lemma 3.3 there exists a finite ortogonal set $S\subset\mathcal{D}_1\cap F^{\perp}$ such that $(S\cup F)^{\perp}=\mathcal{D}_2^{\perp\perp}$. By lemma 3.4 there exists a finite ortogonal set $H\subset\mathcal{D}_1$ such that $F^{\perp}=H^{\perp}$ and $H\subset\sum_{I\in P(n)}A_I^n$. Then $T=H\cup S\cup\mathcal{D}_2$ is a finite fundamental set and so it defines a basic neigbourhood of zero that is the closed interval $[-u,u]$ where $u=V(H\cup S)=(VH)+(VS)$. But $VH\in\sum_{I\in P(n)}A_I^n$ and $VS\in F^{\perp}$ and so, by proposition 2.9 and proposition 2.6,

$$[-u,u]\subset\mathcal{U}(g_1,\dots,g_n;g_{n+1}\cdots g_m)\ \blacksquare$$

The following lemma is corollary 5.2 of [3].

<u>Lemma 3.5.</u> $\mathcal{R}$ is discrete on G iff there exists $\{b_1,\dots,b_n\}\subset\mathcal{D}_2$ such that $G=(\bigvee_{i=1}^{n}b_i)^{\perp}$

As a consequence, under the assumption $\mathcal{D}_1\neq\emptyset$ we have

<u>Corollary 3.6.</u> If G is an archimedean ℓ-group then $\mathcal{R}$ is not discrete.

Proof: Lemma 3.5 and proposition 2.6. ■

<u>Corollary 3.7.</u> Let G be an archimedean finitely founded ℓ-group such that $\mathcal{a}\cup\{0\}$ is inf-semilattice.

If $\mathcal{R}$ is Hausdorff then it is a T-topology.

Corollary 3.8. R is a T-topology on C(R).

Proof: C(R) is an archimedean ℓ-group in which $D_2=\emptyset$ and a is composed of the functions in C(R) whose carrier is a connected set [1]. Therefore, $a\cup\{0\}$ is inf-semilattice. Since each closed set in R is a zero set and each open set is a disjoint union of open intervals we can deduce that C(R) is finitely founded. In [1] it is proved that R is Hausdorff on C(R) and so we can apply corollary 3.7. ■

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