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ON THE LAW OF LARGE NUMBERS FOR CONTINUOUS-TIME MARTINGALES AND APPLICATIONS TO STATISTICS.

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ABSTRACT

In order to develop a general criterion for proving strong consistency of estimators in Statistics of stochastic processes, we study an extension, to the continuous-time case, of the strong law of large numbers for discrete time square integrable martingales (e.g. Neveu, 1965, 1972). Applications to estimation in diffusion models are given.

§1. Introduction.

In statistics of stochastic processes, the problem of estimation in diffusion models has been extensively studied in recent years, using maximum likelihood method for the parametric case (e.g. Brown and Hewitt, 1975; Delebecque and Quadrat, 1975, 1978; Feigin, 1976; Liptser and Shiryayev, 1977, 1978) and kernel me-

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thod for the nonparametric case (e.g. Banon, 1978; Banon and Nguyen, 1978, 1981a, 1981b; Pham, 1978; Geman, 1979; Rao, 1979). Most of these estimators are defined in terms of Ito stochastic integrals and to establish their strong consistency, one is led usually to show that M_t/N_t goes to zero almost surely, as $t\to\infty$, where $(M_t,\ t\geqslant 0)$ is a martingale and $(N_t, t\geqslant 0)$ is some stochastic process. In general, N_t is of the form

$\int_0^t G(s,X_s)ds$ with $G(.,.)\geqslant 0$, and $(X_s,\ s\geqslant 0)$ is an observed process; and it is required that $N_t\to\infty$ almost surely, that is

$$\int_0^\infty G(s,X_s)ds = \infty \quad \text{almost surely} \tag{1.1}$$

For example, in some applications, $N_t = \langle M\rangle_t$, the natural increasing process of a square integrable martingale M_t, then the condition (1.1) is sufficient for $M_t/\langle M\rangle_t\to 0$, almost surely, as $t\to\infty$.

In general, N_t might be different from $\langle M\rangle_t$. The condition (1.1) is no longer sufficient for this purpose, but one can add some new conditions in order that $M_t/N_t\to 0$, almost surely. One way to do this is to find a suitable function g_t such that g_t/N_t is bounded almost surely and sufficient conditions for $M_t/g_t\to 0$, almost surely, $t\to\infty$.

These considerations led us to derive some appropriate forms of the strong law of large numbers for continuous-time square integrable martingales. We shall extend results in Neveu (1965, 1972) on discrete-time case and also the almost sure stability criterion for second order random functions in Loeve (1963), see also Nguyen (1979), to the present case.

As applications, we shall prove the strong consistency of a class of recursive nonparametric estimators of the drift coefficient in the diffusion model, proposed in Pham (1978), and also study some parametric estimation problems. For implementation and simulation studies of these estimators, see Nguyen and Pham

(1981), and Banon and Nguyen (1981b). Some of our results have been announced in Nguyen and Pham (1979).

2. Law of Large Numbers.

In the sequence, all the random variables are real-valued and defined on the same probability space $(\Omega, \mathcal{F}, P)$. $(\mathcal{F}_t, t \geq 0)$ denotes a non decreasing family of right continuous sub-σ-fields of $\mathcal{F}$ (as usual, each $\mathcal{F}_t$ is completed by the P-null sets from $\mathcal{F}$). By increasing process of a right continuous, square integrable martingale M_t, relative to $(\mathcal{F}_t, t \geq 0)$, we mean the increasing process associated with the non-negative submartingale $(M_t^2, t \geq 0)$ in the Doob-Meyer decomposition.

We will need the following lemma (its proof is immediate) in the proof of our main theorem below.

<u>Lemma 1.</u> (Generalized Toeplitz Lemma). Let $(u_t, t \geq 0)$ be a left continuous, non decreasing function tending to infinity as $t \to \infty$, and $(\phi_t, t \geq 0)$ be a function such that $\int_{[0,t)} \phi_s du_s$ exists for all t. If $\phi_s \to 0$ as $t \to \infty$, then

$$\frac{1}{u_t} \int_{[0,t)} \phi_s du_s \to 0 \text{ as } t \to \infty.$$

<u>Theorem 1.</u> Let $(M_t, t \geq 0)$ be a right continuous, locally square integrable martingale* on $(\Omega, \mathcal{F}, P)$, realtive to $(\mathcal{F}_t, t \geq 0)$, with natural increasing process $(\langle M \rangle_t, t \geq 0)$. Let $(u_t, t \geq 0)$ be a non decreasing process, adapted to $(\mathcal{F}_t, t \geq 0)$. Then, as $t \to \infty$, $M_t / u_t \to 0$ almost surely on the set

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* i.e., there exists a sequence of stopping times $\tau_n, \tau_n \uparrow +\infty$ as $n \to +\infty$, such that, for each n, $(M_{t \wedge \tau_n}, t \geq 0)$ is a square integrable martingale.

$$\{u_\infty = \infty\} \cap \left(\bigcup_{n=1}^{\infty} \left\{ \int_n^\infty u_{t-}^{-2} d\langle M\rangle_t < +\infty \right\} \right)$$

Proof: Replacing u_t by u_{t-} which is less than u_t, we can suppose, without loss of generality, that u_t is left continuous. Let $n \geqslant 1$ be such that $u_n > 0$, and m be an integer.

(i) Define: $B_t = \int_{(n,t]} u_s^{-2}\, d\langle M\rangle_s$, $t \geqslant n$.

and $\tau_m = \inf\{t \geqslant m, B_t \geqslant m\}$, with $\tau_m = \infty$ if $B_t < m$ for all t. Since $\{\tau_m \leqslant t\} = \{B_t > m\}$ by the right continuity of $t \to B_t$, and since B_t is predictable (Liptser and Shiryayer, 1977), the stopping time τ_m is predictable. Therefore (Dellacherie and Meyer, 1978, p. 132) there exists a sequence of stopping times $\tau_{m,i}$ such that $\tau_{m,i} < \tau_m, \tau_{m,i} \uparrow \tau_m$ as $i \to \infty$ almost surely. Now, $B_{t \wedge \tau_{m,i}} < m$ so that the stochastic integral $\int_{(n, t\wedge\tau_{m,i}]} u_s^{-1}\, dM_s$ is well-defined and is a square integrable martingale.

Define the process $Z_t^{(m)}$, $n < t < \tau_m$ by the condition

$$Z^{(m)}_{t\wedge\tau_{m,i}} = \int_{(n, t\wedge\tau_{m,i}]} u_s^{-1}\, dM_s.$$

Clearly $Z_t^{(m)}$ is right continuous and admits left limits. We will show now that, almost surely on the set $\{\tau_m = \infty\}$, $Z_t^{(m)}$ also admits a limit when $t \to \infty$. By replacing $\tau_{m,i}$ by and appropriate sub-sequence, one can assume that

$$\underline{P}\{\tau_m = \infty, \tau_{m,i} < i\} < 2^{-i}, \quad i = 1,2,\ldots$$

Therefore, by the Borel-Cantelli lemma, $\tau_{m,i} > i$ for i sufficiently large, almost surely on $\{\tau_m = \infty\}$. Set $\sigma_t = m \vee (t \wedge \tau_{m,[t]+1})$, where $[t]$ is the integer part of t. The family of stopping times σ_t is increasing and bounded by τ_m, hence $Z_\sigma^{(m)}$ is a martingale with natural increasing process B_{σ_t}. Since $B_{\sigma_t} < m$ for all t, it is a uniformly integrable martingale (Neveu, 1965, p. 130), and hence tends to limit as $t\to\infty$. But $\sigma_t=t$ for t sufficiently large, almost surely on $\{\tau_m=\infty\}$, which proves the result.

Define Z_t, $n\leqslant t<\tau=\lim\uparrow\tau_m$ by the condition $Z_{t\wedge\tau_m}=Z_t^{(m)}$. Clearly Z_t is right continuous, admits left limits and $\lim_{t-\infty} Z_t$ exists almost surely on

$$\bigcup_m \{\tau_m=\infty\}=\{\int_n^\infty u_s^{-2}d<M>_s<\infty\}.$$

Thus we have shown that, for almost all w of the above set, $Z_t(w)$ is right continuous, has left limits and has a limit as $t\to\infty$. This implies that $t\to Z_t(w)$ is bounded (Doob, 1953, p. 361).

(ii) Set $t_i^{(k)}=n+(T-n)i/k$, $T>0$, we have

$$u_TZ_T=\sum_{i=1}^{k}\Big((u_{t_i^{(k)}}-u_{t_{i-1}^{(k)}})Z_{t_i^{(k)}}+u_{t_{i-1}^{(k)}}(Z_{t_i^{(k)}}-Z_{t_{i-1}^{(k)}})\Big)$$

$$=\int_n^{T-}Z_t^{(k)}du_t+\int_n^{T}u_t^{(k)}dZ_t$$

where $Z_t^{(k)}=Z_{t_i^{(k)}}$, $u_t^{(k)}=u_{t_{i-1}^{(k)}}$ for $t_{i-1}^{(k)}\leqslant t<t_i^{(k)}$.

Since $Z_t^{(k)}\to Z_{t+}$ and $Z_t^{(k)}$ is bounded for all t, all k, almost surely, by Lebesgue dominated convergence Theorem, as $k\to\infty$

$$\int_n^{T-}Z_t^{(k)}du_t\to\int_n^{T-}Z_{t+}du_t.$$

In the same way, since $u_t^{(k)}\to u_t$ and $u_t^{(k)}$ bounded for all $t<T$, k, as $k\to\infty$

$$\int_n^{T}u_t^{(k)}dZ_t\to\int_n^{T}u_tdZ_t=\int_n^{T}u_t(u_t^{-1}dM_t)=\int_n^{T}dM_t.$$

Therefore

$$\lim_{T\to\infty} u_T^{-1} M_T =$$

$$\lim_{T\to\infty} \{Z_T - u_T^{-1} \int_n^{T-} Z_{t+} du_t\} = \lim_{T\to\infty} \{(Z_T - Z_{\infty-}) - u_T^{-1} \int_n^{T-} (Z_{t+} - Z_{\infty-}) du_t\}.$$

So by lemma 1, $M_T/u_T \to 0$ almost surely, on the set $\{\lim\uparrow u_T = \infty\}$, as $T \to \infty$. This completes the proof of the theorem.

<u>Corollary 1.</u> Let $(M_t, t \geq 0)$ be a right continuous, locally square integrable martingale relative to $(\mathcal{F}_t, t \geq 0)$, with natural increasing process $(\langle M\rangle_t, t \geq 0)$, and let $(u_t, t \geq 0)$ be a non decreasing process, adapted to $(\mathcal{F}_t, t \geq 0)$. If for some $n \geq 1$,

$$E \int_n^\infty u_{s-}^{-2} \, d\langle M\rangle_s < \infty$$

then $u_t^{-1} M_t \to 0$, almost surely on the set $\{u_\infty = \infty\}$, as $t \to \infty$.

<u>Corollary 2.</u> Let $(M_t, t \geq 0)$ be a right continuous, square integrable martingale relative to $(\mathcal{F}_t, t \geq 0)$, with natural increasing process $(\langle M\rangle_t, t \geq 0)$, and $(g_t, t \geq 0)$ be a non decreasing function tending to infinity as $t \to \infty$ such that for some $n \geq 1$

$$\int_n^\infty g_{t-}^{-2} \, d\gamma_t < \infty \text{ where } \gamma_t = E\langle M\rangle_t$$

then $u_t^{-1} M_t \to 0$, almost surely as $t \to \infty$.

Proof: It follows simply from the fact that, if $(g_t, t \geq 0)$ is non random, then

$$E \int_n^\infty g_{t-}^{-2} \, d\langle M\rangle_t = \int_n^\infty g_{t-}^{-2} \, dE\langle M\rangle_t .$$

<u>Corollary 3.</u> Let $(M_t, t \geq 0)$ be a right continuous, square integrable martingale relative to $(\mathcal{F}_t, t \geq 0)$, and $(g_t, t \geq 0)$ be a non

negative increasing function tending to infinity as $t \to \infty$. If there exists a sequence of non negative real numbers $(\alpha_n, n \geq 1)$ increasing, tending to infinity and such that

$$\sum_{m=1}^{\infty} \gamma_{\alpha_{m+1}} / g^2_{\alpha_m} < +\infty, \text{ where } \gamma_t = E\langle M\rangle_t,$$

then $g_t^{-1} M_t \to 0$, almost surely, $t \to \infty$.

In particular, the same result holds if for some $a>0$,

$$\sum_{m=1}^{\infty} \gamma_{(m+1)^a} / g^2_{m^a} < +\infty.$$

Proof: The result follows from the Corollary 2 by observing that

$$\int_{\alpha^2}^{\infty} g_{t-}^{-2} d\gamma_t \leq \sum_{m=1}^{\infty} (\gamma_{\alpha_{m+1}} - \gamma_{\alpha_m}) / g^2_{\alpha_m} < \infty.$$

Remark. This corollary 3 is an extension of the almost sure stability criterion in Loève (1963, p. 486), see also Nguyen (1979). To illustrate the extension of the proof used in Loève, the corollary 3 can be proved directly as follows:

For $\alpha_m \leq t < \alpha_{m+1}$ we have:

$$g^{-1}_{\alpha_m} M_t = g^{-1}_{\alpha_m} M_{\alpha_m} + Z(\alpha_m, t)$$

where $Z(\alpha_m, t) = g^{-1}_{\alpha_m} (M_t - M_{\alpha_m})$. Set $U(\alpha_m) = \sup_{\alpha_m \leq t \leq \alpha_{m+1}} |Z(\alpha_m, t)|^2$.

Since $\{|Z(\alpha_m, t)|^2, t \in [\alpha_m, \alpha_{m+1}]\}$ is a sub-martingale, we have (Neveu, 1965, p. 133), for $c>0$:

$$cP\{U(\alpha_m) > c\} \leq E|Z(\alpha_m, \alpha_{m+1})|^2 = g^{-2}_{\alpha_m} (\gamma_{\alpha_{m+1}} - \gamma_{\alpha_m}).$$

Since $\gamma_{\alpha_{m+1}} > \gamma_{\alpha_m}$, the condition (2.1) implies that $\sum_{m=1}^{\infty} P\, U(\alpha_m)>c\}<\infty$, and hence by Borel-Cantelli lemma, $U(\alpha_m) \to 0$ almost surely as $m \to \infty$. On the other hand,

$$\sum_{m=1}^{\infty} E\,|g_{\alpha_m}^{-1} M_{\alpha_m}|^2 = \sum_{m=1}^{\infty} g_{\alpha_m}^{-2} \gamma_{\alpha_m} < \infty \text{ by (2.1)},$$

and therefore, by Borel-Cantelli lemma again, $g_{\alpha_m}^{-1} M_{\alpha_m} \to 0$ almost surely, as $m \to \infty$.

We obtain the desired result by observing that $|g_t^{-1} M_t| \leq |g_{\alpha_m}^{-1} M_t|$.

<u>Corollary 4.</u> In the notation of the corollary 3, if for some $\delta > 1$, $\alpha > 0$ we have $\gamma_{\delta t}/g_t^2 = 0(t^{-\alpha})$, $t \to \infty$, then $g_t^{-1} M_t \to 0$ almost surely, $t \to \infty$. The same conclusion holds if $g_t \sim t^{\beta}$, $\gamma_t = 0(t^{2\beta-\alpha})$, $t \to \infty$, with $\alpha > 0$, $\beta > 0$.

<u>Corollary 5.</u> In the notation of the corollary 4, and suppose that there exists a non decreasing measurable function $f:(0,\infty) \to R^+$ such that $f(\gamma_t) = 0(g_t^2)$, $t \to \infty$, and

$$\int_n^{\infty} f^{-1}(u)\,du < \infty \text{ for some } n \geq 1,$$

then $g_t^{-1} M_t \to 0$ almost surely as $t \to \infty$.

Proof: It is enough to check that $\int_u^{\infty} g_t^{-2}\, d\gamma_t < \infty$ for some a. We have $f(\gamma_t) g_t^{-2} \leq c$ for t sufficiently large, $t \geq a$, say. Therefore

$$\int_a^{\infty} g_t^{-2}\, d\gamma_t = \int_a^{\infty} f(\gamma_t) g_t^{-2} f^{-1}(\gamma_t)\, d\gamma_t \leq c \int_a^{\infty} f^{-1}(\gamma_t)\, d\gamma_t .$$

Set $\alpha(x) = \inf\{t:\gamma_t \geq x\}$, then $\alpha(x) \leq t$ is equivalent to

$\gamma_t \geqslant x$, and hence the image of the measure dx by α is the measure $d\gamma_t$. Thus:

$$\int_a^\infty f^{-1}(\gamma_t)\,d\gamma_t = \int_{\{\alpha(x)\geqslant a\}} f^{-1}(\gamma_{\alpha(x)})\,dx \leqslant \int_{\gamma_{a^-}}^\infty f^{-1}(x)\,dx$$

since $\alpha(x)\geqslant a$ is equivalent to $x\geqslant \gamma_{a^-}$ and $\gamma_{\alpha(x)} \geqslant x$, the result follows.

As an application of corollary 5, we have

Corollary 6. If for some $\varepsilon > 0$, $\gamma_t(\mathrm{Log}\,\gamma_t)^{1+\varepsilon} = 0(g_t^2)$, in particular if $\gamma_t = 0(g_t^{2-\varepsilon})$ as $t \to \infty$, then $M_t/g_t \to 0$, almost surely, $t \to \infty$.

Corollary 6 has an interesting interpretation, it says that if g_t converges to infinity more rapidly than $\sqrt{\gamma_t}$, in a certain sense, then $M_t/g_t \to 0$ almost surely as $t \to \infty$. Note that $\sqrt{\gamma_t}$ is the L^2 norm of M_t.

Corollary 7. With the same notation as in Corollary 3 and suppose that there exists a measurable, non-decreasing function $f\colon (0,\infty) \to R^+$ such that $f(\langle M\rangle_t) = 0(u_t^2)$ as $t \to \infty$, almost surely, and

$$\int_a^\infty f^{-1}(u)\,du < +\infty$$

for some $a\geqslant 0$. Then $M_t/u_t \to 0$ almost surely on $\{u_{\infty^-} = +\infty)$, as $t\to\infty$.

Corollary 8. If for some $\varepsilon > 0$, $\langle M\rangle_t(\mathrm{Log}\langle M\rangle_t)^{1+\varepsilon} = 0(u_t^2)$, in particular if $\langle M\rangle_t = 0(u_t^{2-\varepsilon})$, as $t \to \infty$, almost surely, then $M_t/u_t \to 0$ almost surely on $\{u_{\infty^-} = +\infty\}$ as $t \to \infty$.

The proof of these results is the same as that of corollaries 5 and 6, using the fact that $t \to \langle M\rangle_t$ is right-continuous.

Remark. If M_t has continuous sample functions and $<M>_\infty = \infty$, almost surely, then M_{τ_t}, $t \geq 0$, where

$$\tau_t = \operatorname{Inf}\{u: <M>_u \geq t\}$$

is a Brownian motion. Thus by the law of iterated logarithm

$$\limsup_{t \to \infty} \left(M_{\tau_t} \Big/ \sqrt{2t \log \log t}\right) = 1$$

or equivalently

$$\limsup_{u \to \infty} \left[M_u / (2<M>_u \log \log <M>_u)^{1/2}\right] = 1$$

Hence $M_u <M>_u^{-1/2} [\log <M>_u]^{-(1+\varepsilon)/2} \to 0$, as $u \to \infty$, almost surely. Corollary 8 says that this result still holds for non continuous martingales.

§3. Applications.

3.1. Consider the estimation of parameter in parametric model of diffusion process

$$dX_t = \mu_\theta(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0$$

θ being the unknown parameter to be estimated ($\theta \in \text{Ⓗ} \subset R^d$). Now, if the function $\mu_\theta(\cdot)$ is linear in θ, then the log likelihood function is quadratic in θ and the maximum likelihood estimate θ_T of θ, based on X_t, $0 \leq t \leq T$, satisfies:

$$0 = \ell_T(\theta_T) = I_T(\theta_T - \theta) + \ell_T(\theta)$$

where $\ell_T(\theta)$ and I_T are the vector of first derivatives and the matrix of second derivatives of $L_T(\theta)$, the log likelihood function based on X_t, $0 \leqslant t \leqslant T$. Thus $\theta_T - \theta = I_T^{-1} \ell_T(\theta)$ where $\ell_T(\theta)$ is in general a square integrable martingale (Feigin, 1976). Thus to show the strong consistency of θ_T, we are led to show the almost sure convergence to 0 as $T \to \infty$ of random variables of the form $M_T / \mathcal{N}_T$ where $M_t, t \geqslant 0$ is a martingale and $\mathcal{N}_t, t \geqslant 0$ is some random process. In the scalar case, that is the case $d = 1$, $\mathcal{N}_t = I_t = \langle \ell(\theta) \rangle_t = \langle M \rangle_t$, so by corollary 6, a sufficient condition for the strong consistency of θ_T is $I_{\infty -} = \infty$, almost surely (see Liptser & Shiryayev, 1978, p.206, Feigin, 1976).

However, in the above example $\mathcal{N}_t$ would be different from $\langle M \rangle_t$ in the vector case ($d > 1$). The same situation occurs in the non parametric estimation of $\mu(x)$ of the model

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \tag{3.1}$$

where the proposed estimate is (Pham, 1978)

$$\mu_T(x) = \int_0^T H_t K\left(\frac{X_t - x}{h_t}\right) dX_t \Big/ \int_0^T H_t K\left(\frac{X_t - x}{h_t}\right) dt \tag{3.2}$$

Here K is some probability density, $H_t \geqslant 0, h_t \geqslant 0$ with $h_t \downarrow 0$ as $t \to \infty$ and

$$g_T = \int_0^T H_t h_t dt < +\infty \tag{3.3}$$

tending to ∞ as $T \to \infty$. Using (3.1), it is seen that the estimate (3.2) is of the form

$$\mathcal{N}_T(x) = \frac{a_T(x)}{\mathcal{N}_T(x)} + \frac{b_T(x)}{\mathcal{N}_T(x)}$$

where $\mathcal{N}_T(x)$ is the denominator of (3.2) and

$$a_T(x) = \int_0^T H_t K\left(\frac{X_t - x}{h_t}\right) \mu(X_t) dt \tag{3.4}$$

(3.5) $$b_T(x) = \int_0^T H_t K\left(\frac{X_t - x}{h_t}\right)\sigma(X_t)\,dW_t$$

Since $b_T, T \geqslant 0$ is a martingale, we are led to the same problem as above.

3.2. We now return to the first example in §3.1 and consider the simple model

$$dX_t = \theta X_t dt + dW_t, t \geqslant 0, \quad \theta \in R.$$

Here the maximum likelihood estimate θ_T of θ satisfies

$$\theta_T - \theta = \int_0^T X_t dW_t \Big/ \int_0^T X_t^2 dt = M_T/\langle M\rangle_T, \text{ say}$$

So all we need is to show that $\langle M\rangle_{\infty-} = \infty$. For this we shall find a suitable function g_t tending to infinity, such that $g_T/\langle M\rangle_T$ tend to a finite limit almost surely, as $T \to \infty$.

Suppose $\theta < 0$, by ergodicity, we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T X_t^2 dt = \lim_{t\to\infty} E(X_t^2) > 0$$

(see Feigin, 1976), therefore it suffices to choose $g_t = t$.

Suppose $\theta > 0$, then (Feigin, 1976)

$$X_t^2 = e^{2\theta t}(Z + X_0)^2 + 0(e^{2\theta t}),$$

almost surely as $t \to \infty$, where Z is independent of X_0 and different from 0 almost surely. Therefore it suffices to choose $g_t = e^{2\theta t}$ to have

$$g_T^{-1} \int_0^T X_t^2 dt \to 2\theta(Z + X_0)^2$$

by the lemma 2.

Note that we can also show, by corollary 1 or 4, that $g_T^{-1} \int_0^T X_t dW_t \to 0$ almost surely, as $T \to \infty$ since $\gamma_t = 0(t)$ if $\theta < 0$ and $0(e^{2\theta t})$ if $\theta > 0$. Thus one can obtain the result without using corollary 6.

Finally, the case $\theta = 0$ needs a special treatment. We have $X_t = X_0 + W_t$, hence to show that $\int_0^{+\infty} X_t^2 dt = +\infty$ almost surely, we need only to verify that $\int_0^{+\infty} (x_0 + W_t)^2 dt = +\infty$ almost surely. Now let $\Lambda(t,x)$ be the Brownian local time (Freedman, 1971, p. 138), we have

$$\int_0^T (x_0 + W_t)^2 dt = \int_{-\infty}^{+\infty} (x_0 + x)^2 \, \Lambda \, (T,x) dx$$

and since $\Lambda(T,x)$ increases to $+\infty$, as $T \to +\infty$, for almost every x, almost surely, the result follows. See also Nguyen and Pham (1979).

Remark. Using Fubini's theorem, it is easy to show that if $\Lambda(+\infty,x) = +\infty$, a.s., $\forall x$, then $\Lambda(+\infty,x) = +\infty$, for almost every x, almost surely.

3.3. We now show the strong consistency of the estimate (3.2). It is known that under suitable conditions, there is a stationary solution $X_t, t \geqslant 0$, of (3.1). So we suppose that the observed process X_t is stationary and we denote by f the common density of the X_t. It is interesting to note that the recursive Kernel type estimators of $f(x)$ based on $X_t, t \in [0,T]$, is of the form (Banon, 1978, Nguyen, 1979)

$$f_T(x) = \frac{1}{g_T} \int_0^T H_t K\left(\frac{X_t - x}{h_t}\right) dt$$

where g_T is given by (3.3). So, the estimate $\mu_T(x)$ can be written as

$$\mu_T(x) = \frac{\alpha_T(x)}{f_T(x)} + \frac{\beta_T(x)}{f_T(x)}$$

where $\alpha_T(x) = a_T(x)/g_T, \beta_T(x) = b_T(x)/g_T, a_T(x)$ and $b_T(x)$ being defined in (3.4), (3.5).

In Banon and Nguyen (1978), it is shown that $f_T(x)$ is strongly consistent. The same technique can be used here to show that $\alpha_T(x)$ converges strongly to $\mu(x)f(x)$. The proof is reproduced briefly here.

Lemma 2. If f is continuous and either K has compact support or f is bounded, then $Ef_T(x) \to f(x)$ as $T \to \infty$. More generally, if ψf is continuous and either K has compact support or ψf bounded, then

$$E\{g_T^{-1} \int_0^T H_t K(\frac{X_t - x}{h_t}) \psi(X_t)dt\}$$

tends to $\psi(x)f(x)$ as $T \to \infty$.

Proof. The first part of the lemma has been shown in Nguyen (1979, proposition 1), using Toeplitz lemma and the fact that

$$\int K(y)\{f(x + h_t y) - f(x)\}dy \to 0$$

as $t \to \infty$. The second part is an easy generalization and has been proved in Pham (1978).

From now on we shall suppose that the X_t process satisfies the condition G_2 of Rosenblatt (1970), namely

$$E[E\{\phi(X_{t+s})|X_t\}]^2 \leqslant \delta^s \; E\phi^2(X_t)$$

for some $s > 0, 0 < \delta < 1$ and all ϕ such that $E\phi(X_t) = 0$. Using the Markov property of the X_t process, it can be shown (Banon & Nguyen, 1978) that G_2 implies:

$$|E\{\phi(X_{t+s})\psi(X_t)\}| \leqslant C\delta^s\{E\phi^2(X_t)E\psi^2(X_t)\}^{1/2}$$

where C is some constant and ϕ, ψ are such that $E\phi(X_t)=E\psi(X_t)=0$.

Now, let us put

$$Z_t = H_t K(\frac{X_t - x}{h_t})\,\psi(X_t)$$

$$\Gamma_{st} = \operatorname{cov}(Z_s, Z_t).$$

Then the above condition means that $|\Gamma_{s,t}| \leqslant C\delta^{|t-s|}(\Gamma_{ss}\Gamma_{tt})^{1/2}$. But if ψ^2 satisfied the condition of lemma 2:

$$\Gamma_{t,t} \leqslant E\, Z_t^2$$

$$= H_t^2 h_t \int K(y)\psi^2(x + h_t y) f(x + h_t y)\,dy$$

$$= O(H_t^2 h_t), \quad t \to \infty$$

and hence $|\Gamma_{s,t}| \leqslant \hat{C} H_x H_t \sqrt{h_s h_t}\,\delta^{|t-s|}$, where $\hat{C}$ is some constant.

We now use the almost sure convergence criterion in Nguyen (1979), which we recall here.

<u>Theorem 2.</u> "Let $Z_t, t \geqslant 0$ be a measurable second order process and $g_t, t \geqslant 0$ a positive measurable function with $g_t > 0$ for $t > 0$ and $g_t \to \infty$ as $t \to \infty$. Let

$$U_t = \frac{1}{g_t}\int_0^T (Z_s - E\, Z_s)\,ds$$

Suppose that

(i) $g_t \sim t^\beta$, $t \to \infty$ with $0 < \beta < 1$

(ii) For sufficiently large t, $\operatorname{var}(Z_t) \leqslant c$ and

$$\frac{1}{t^{2}\beta}\left|\int_0^t \int_0^t \mathrm{cov}\{Z_s, Z_{s'}\} ds\, ds'\right| \leqslant \frac{d}{t^{\gamma\beta}}$$

where c, d, γ, β are strictly positive constants with $\gamma\beta > 2(1-\beta)$.

Then $U_t \to 0$ as $t \to \infty$, almost surely.

Theorem 3. Suppose that

(i) $f, \mu f, \mu^2 f$ are continuous and either K has compact support or $f, \mu f, \mu^2 f$ are bounded

(ii) Either a) or b) below is satisfied

a) H_t is bounded and $g_t \sim t^{\beta}$, with $2/3 < \beta < 1$

b) $H_t = 0(h_t^{1/2})$ and $g_t \sim t^{\beta}$ with $3/4 < \beta < 1$.

Then $f_T(x) \to f(x)$ and $\alpha_T(x) \to \mu(x) f(x)$ almost surely as $T \to \infty$.

Proof. By lemma 2, $Ef_T(x) \to f(x)$ and $E\alpha_T(x) \to \mu(x)f(x)$. To show that $f_T(x) - Ef_T(x)$ and $\alpha_T(x) - E\alpha_T(x)$ tend to zero almost surely, we shall use the almost sure convergence criterion in Theorem 4. Set

$$Z_t = H_t K\left(\frac{X_t - x}{h_t}\right)\psi(X_t)$$

where $\psi(\cdot) = 1$ or $\psi(\cdot) = \mu(\cdot)$. Then we have seen that for large s,t

$$|\mathrm{cov}\{Z_s, Z_t\}| \leqslant \mathrm{const}.H_s H_t \sqrt{h_s h_t}\ \delta^{|t-s|}$$

So, by a) or b), $\mathrm{var}(Z_t)$ is bounded as $t \to \infty$ and

$$\left|\int_0^T \int_0^T \mathrm{cov}\{Z_s, Z_t\} ds\,dt\right| \leqslant 2\int_0^\infty \left[\int_0^\infty |\mathrm{cov}\{Z_t, Z_{t+u}\}| du\right] dt$$

which is $O(g_T)$ under a) and $O(T)$ under b), as $T\to\infty$. Therefore conditions (ii) of Theorem 1 is satisfied with $\gamma = 1$ in case a) and $\gamma = 2 - 1/\beta$ in case b). The result follows.

Corollary 9. Under the condition of the Theorem, if $f(x)>0$ then $\alpha_T(x)/f_T(x)\to \mu(x)$ almost surely as $T\to\infty$.

It remains to show that $\beta_T(x)\to 0$ almost surely as $T\to\infty$. Now $\beta_T(x) = b_T(x)/g_T$ where $b_T(x)$ is given in (3.5). Set

$$\gamma_t = E\langle b\rangle_t = E[\int_0^t H_s^2 K(\frac{X_s - x}{h_s})\sigma(X_s)ds].$$

Suppose that the function $\sigma(\cdot)$ satisfy the assumption of lemma 2, then the same proof as that of this lemma shows that

$$\gamma_t = O(\int_0^t H_s^2 h_s ds)$$

and hence $\gamma_t = O(g_t)$ under the assumption a) and $\gamma_t = O(T)$ under the assumption b), of Theorem 3. The application of Carollary 4 or Corollary 6 then shows that $b_T(x)/g_T(x)\to 0$ almost surely as $T\to\infty$, provided that $g_t\sim t^\beta$ with $\beta > 1/2$. We thus obtain

Theorem 4. Suppose that σf is continuous and either K has compact support of σf is bounded. Then under the assumption a) or b) of Theorem 3, $\beta_T(x)\to 0$ almost surely as $T\to\infty$.

Corollary 10. Under the assumptions of Theorems 3 and 4 suppose that $f(x) > 0$, then $\mu_T(x)$, given by (3.2) is a strongly consistent estimator of $\mu(x)$.

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