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ON m-DIMENSIONAL STOCHASTIC PROCESSES IN BANACH SPACES

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ABSTRACT

In the present paper the authors prove a weak law of large numbers for multidimensional processes of random elements by means of the random weighting. The results obtained generalize those of Padgett and Taylor.

1. In the last decade Taylor, Padgett and Wei [2], [3], [4], [5] and other authors have promoted the study of the convergence of sums of random elements, i.e. random variables in Banach spaces, and in particular they proved laws of large numbers and related theorems. In this short note we generalize some results by Padgett and Taylor to the case of multidimensional processes of random elements and particularly we prove a weak law of large numbers by means of a suitable random weighting.

Key Words.
multidimensional processes, weak law of large numbers, random weighting, random elements.

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We start with the standard definitions and notations as in the papers of cited authors.

Let W be a Banach space with norm $\|\cdot\|$ and let B(W) the Borel σ-field generated by open subsets of W.

Let (Ω, A, P) be a probability space and let V be a function from Ω into W. If $V^{-1}(B) \in A$ for every $B \in B(W)$, then V is said to be a random element in W.

We consider a multidimensional random process (Wong [6]) of random elements as a family $\{X_t\}$ of random elements on W, with index t belonging to the positive integer lattice N_0^m of R^m.

Let $\{U_k, k \in N\}$ be a sequence of finite subsets of N_0^m which is a partition of N_0^m, the process $\{X_t\}$ is said to be identically distributed with respect to the partition U_k, if

$$P(\sum_{t \in U_k} X_t) = P(\sum_{t \in U_n} X_t) \text{ for all } n,k \text{ such that } n \neq k \text{ and}$$

for all $B \in B(W)$.

(e.g. U_k is $\{t \in N_0^m: |t| = k\}$, where $|t| = t_1 + \ldots + t_n$)

A process $\{X_t\}$ is said to be independent if every finite subset is independent. From theorem 3.3.2 of Chung [1] and lemma 2.3.5 of Padgett and Taylor [2], follows that the sequence $\{\sum_{t \in U_k} X_t, K \in N\}$ is independent for all partitions U_k.

Finally, let $\{a_k\}$ be a sequence of random variables, if we assume the weighted process $\{X_t a_k, t \in U_k\}$ is independent and identically distributed with respect to U_k, we are able to prove a weak law of large numbers for the original process X_t.

2. We first state the following useful lemma:

Lemma: Let (Ω, A, P) be a probability space and let X_t an independent and identically distributed (with respect to U_k) m-dimensio nal process of random elements in the Banach separable space W and let $\{b_{nk};n,k\in N\}$ be a double array of random real variables.

Set $V_k = \sum_{t\in U_k} X_t$, we suppose that

a) $\|V_k\|$ is integrable, for all $k\in N$,

b) $\lim_n \sup \sum_{k=1}^{n} |b_{nk}|^2 < +\infty$ a.s. in Ω,

then

$$\lim_n n^{-1} \sum_{k=1}^{n} b_{nk} V_k = 0 \quad \text{a.s. in } \Omega .$$

<u>Proof:</u> The proof is the same of that of theorem 3 of [5] and is mitted for the sake of brevity.

Now, we are in condition to prove the result of the paper

<u>Theorem:</u> Let (Ω,A,P) be a probability space and let W a separable Banach space having a Schauder basis$^{(*)}$.

Let X_t be the above defined m-dimensional process of random elements in W and let $\{a_k\}$ a sequence of real random variables.

Set $V_k = \sum_{t\in U_k} X_t$, we assume that

c) the weighted process $\{a_k X_t, t\in U_k\}$ is independent and identically distributed with respect to U_k,

d) $\lim_n \sup \sum_{k=1}^{n} |a_k^{-1} - 1|^2 < +\infty$ a.s. in Ω

and $a_1 = c$ $(c\in R-\{0\})$ a.s. in Ω,

e) $E(\|V_1\|) < +\infty$ and $E(a_k V_k) = E(a_1 V_1)$ for all $k\in N$,

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(*) A sequence $\{b_n\}\subset W$ is said to be a Schauder basis for W if there exists a unique sequence of scalars $\{t_n\}$ such that $w = \sum_k t_k b_k$ for all $w\in W$.

then the following conditions are equivalent

f) the sequence of random real variables $\|n^{-1}\sum_{k=1}^{n}(V_k-E(V_1))\|$ tends to zero in probability.

g) the sequence of random real variables $n^{-1}\sum_{k=1}^{n} b_{nk}\ f_i(V_k-E(V_1))$, where f_n is a sequence of coordinate functionals, linear and continuous (e.g. a sequence defined by letting $f_n(w)=t_n$, for all $w\in W$), tends to zero in probability.

Proof: We denote by $\mathcal{L}$ the set of sequences $\{Z_k,\ k\geq 1\}$ of random elements on (Ω,A,P) into W such that the following conditions are equivalent to one another

h) the sequence of random real variables $\|n^{-1}\sum_{k=1}^{n} Z_k\|$ tends to zero in probability,

i) for every coordinate functionals f_i the sequence of random real variables $n^{-1}\sum_{k=1}^{n} f_i(Z_k)$ tends to zero in probability.

It is evident that $\mathcal{L}$ is a vector space. We shall prove that the sequence $\{(V_k - E(V_1)),\ k\geq 1\}$ belongs to $\mathcal{L}$.

First, we assume $a_1 = 1$. We have

$$V_k - E(V_1) = a_k^{-1}a_k\ V_k - a_k\ V_k + a_k\ V_k-E(a_1V_1)=$$

$$(a_k^{-1} -1)a_k\ V_k+a_k\ V_k -E(a_k\ V_k).$$

By applying theorem 5.1.1 of Padgett-Taylor [2] to the weighted process, the difference $(a_k\ V_k - E(a_kV_k))$ belongs to $\mathcal{L}$.

On the other hand, also the sequence $\{(a_k^{-1} - 1)a_k\ V_k\}$ belongs to $\mathcal{L}$: in fact, putting $b_{nk} = a_k^{-1} - 1$ in the lemma, we have that the sequence of the real random variables

$$\|n^{-1}\sum_{k=1}^{n}(a_k^{-1}-1)\ a_k\ V_k\|$$

tends to zero a.s. in Ω.

Now, if $a_1 = c \in R-\{0\}$, then from condition e) we can derive $E(V_1) = E(c^{-1}\, a_k\, V_k)$.

By employing the result of lemma 2.3.3 of Taylor [4] the process $\{c^{-1}\, a_{|t|} X_t,\ t \in N_0^m\}$ is independent and identically distributed. Hence the proof.

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