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ON THE MEASUREMENT OF THE ACTIVITY OF A RADIOACTIVE SOURCE AND A RELATED STOCHASTIC PROCESS

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ABSTRACT

A method is presented to compute the activity of a radioactive source. The principle of the method is based on the "tuning" of β, the time constant of the RC circuit of the detector with λ being the rate of emission of the source, using a statistical argument.

The stochastic process involved refers to the distribution of the following random voltage:

$$V_t = \sum_{o<t_i \leq t} Y_i \, e^{-\beta(t-t_i)}$$

where the t_i are Poisson dates of emission and the Y_i are random or deterministic pulse heights. The case of Y_i gamma distributed is investigated. This method could replace a crude counting for the case of a very intense source where this kind of measurement would be delicate due to the problem of rapid data acquisition.

I) Introduction.

From the results presented in [1], see also [11], we propose the following experiment. Particles are emitted according to a

Poisson process with rate λ, these particles are counted by a detector included in an electrical circuit with a time constant

$$\beta = \frac{1}{RC}$$

The calibrated pulses of the detector (W) delivered at Poisson random dates are recorded during the time interval $(0,t]$.

Then, the voltage measured at time t is:

$$V_t = \sum_{o<t_i \leq t} U_i$$

where $U_i = \exp(-\beta(t-t_i))$,

and the t_i are Poisson random variables representing the dates at which the particles are detected.

In the stationary case for which $t\to\infty$, the conditional distribution

$$\Pr [V_\infty \leq v \,/\, V_\infty \leq 1]$$

is uniform for $v \in [0,1]$, see [6], where U_i, V are expressed in unit of calibrated pulse W. In practice, if t is sufficiently large, i.e. equal to n/λ where $n \sim 10$ and if we reject all the pulses $V_{n/\lambda}>1$, a sample of pulses $V_{n/\lambda}$ given by an amplitudes selector will be asymptotically $(n\to\infty)$ uniformly distributed. (see (8) for the distribution of $\Pr(v_n \leqslant 1)$).

II) Experiment.

(see scheme)

The experimental choice will cover the following situation:

1) The radioactive source (whose emission is Poisson distributed),

its quality (α,β,γ), its emission rate λ, the value of the solid angle used to perform the measurements. A possible application could be the measurement of the intensity of a X-rays beam.

2) The detector used to deliver the calibrated pulses of W volts (Geiger, p-n diode, scintillator,...) its efficiency and its associated electronic circuits and particularly the variable time constant of the circuit β, such that

$$\beta = \frac{1}{RC}$$

3) The value n sufficient to consider the behaviour of $V_{n/\lambda}$ as stationary. (i.e. for a sufficient time of storage of the pulses).

4) The value of the calibrated pulses W such that, when discriminated, the pulses inferior to W analyzed by a multichannel amplitudes analyzer are not too much perturbed by the noise of the device, discriminated also at a threshold $W_o \ll W$.

III) Analysis.

After having collected in the multichannel analyzer a sufficiently large number of pulses $V_{n/\lambda}$, we will test using a non-parametric test [2], whether the conditional distribution is uniformly distributed as predicted by the theory. If $V_{n/\lambda}$ is not uniform ($\beta \neq \lambda$):

1) <u>NON UNIFORM CASE:</u>

If λ and β are different, the conditional distribution is given by: (see [6])

$$\Pr\,[\,v_\infty \leq v \;/\; v_\infty \leq 1\,] = v^{\lambda/\beta} \;, \quad v \in [0,1]$$

and $$\Pr[v_\infty \leqq 1] = \frac{e^{-\lambda/\beta\gamma}}{\Gamma(\frac{\lambda}{\beta}+1)}$$

where γ is the Euler constant (= 0.577).

$\Pr[v_\infty > 1]$ represents the percentage of rejected pulses by the amplitude discriminator. In that case λ can be easily determined from the sample of pulses using the conditional average theoretically equal to:

$$\mu = \frac{\lambda/\beta}{\lambda/\beta+1}$$

and the estimated value:

$$\hat{\lambda} = \beta \frac{\hat{\mu}}{1-\hat{\mu}}$$

where $\hat{\mu}$ is the observed value of the conditional mean. Then the conditional distribution could be tested using [4].

2) UNCALIBRATED PULSES.

If the pulses delivered by the detector are not calibrated but rather randomly distributed with a distribution stochastically independent of the arrival dates, the Laplace transform of the probability density of the stationary signal V_∞ is given by

$$\tilde{f}(s) = \exp\{\frac{\lambda}{\beta}\int_o^s \frac{\tilde{h}(\xi) - 1}{\xi} d\xi\}$$

where $\tilde{h}$ is the Laplace transform of the pulses heights distribution [6]. (see also ref.(18)).

Let us take an example, suppose h is gamma distributed with a parameter δ:

$$h(x) = \frac{e^{-x}\, x^{\delta-1}}{\Gamma(\delta)} \quad , \quad x \geqq 0 \quad , \delta > 0$$

Then
$$\tilde{h}(\xi) = (1+\xi)^{-\delta}$$

a) CASE $\delta = 1$ see [1].

b) CASE $\delta = 2$.

We get for $\tilde{f}$ after straight forward integration:

$$\tilde{f}(s) = \frac{e^{-\lambda/\beta}}{(1+s)^{\lambda/\beta}} \exp\{\lambda/\beta(1+s)\}.$$

The corresponding probability density for V_∞ is:

$$f(t) = e^{-\lambda/\beta} e^{-t} \left(\frac{\beta t}{\lambda}\right)^{(\lambda/\beta-1)/2} I_{\lambda/\beta^{-1}}(2\sqrt{\lambda t/\beta}), \quad t \geq 0.$$

where I_α is a modified Bessel function.

c) CASE $\delta = 0.5$.

We get for $\tilde{f}$ after straight forward integration:

$$\tilde{f}(s) = \frac{4^{\lambda/\beta}}{(1+\sqrt{1+s})^{2\lambda/\beta}}$$

The corresponding probability density for V_∞ is:

$$f(t) = \sqrt{\frac{2}{\Pi}}\, 2^{3\lambda/\beta} \frac{\lambda}{\beta} e^{-t/2}\, t^{\lambda/\beta^{-1}}\, D_{-(1+2\lambda/\beta)}(\sqrt{2}t), \quad t \geq 0$$

where D_α is the parabolic cylinder function. This parabolic cylinder density appears for instance in [5].

In these cases the value λ can be deduced from the expectation of these densities in the same way as previously: i.e. replacing in the theoretical expression the mean by the observed mean.

d) CASE $\delta = 1.5$.

We get for $\tilde{f}$ after straightforward integration:

$$\tilde{f}(s) = \frac{4^{\lambda/\beta}}{(1+\sqrt{1+s})} 2\gamma/\beta \exp\{2\frac{\lambda}{\beta}(\frac{1}{\sqrt{s+1}} - 1)\}.$$

The corresponding probability density for V_∞ is:

$$f(t) = \frac{4^{\lambda/\beta} e^{-2\lambda/\beta}}{2\sqrt{\eta}\Gamma(2\gamma/\beta)\ t^{3/2}} [\int_0^\infty \xi_e^{\lambda+1/\beta} e^{-\xi^2/4t} \Phi_3(2^{\lambda}/\beta 2^{\lambda}/\beta;-\xi;2\frac{\lambda}{\beta}\xi)d$$

$$- \int_0^\infty \xi e^{-\xi^2/4t} \int_0^\xi J_1(\eta)(\xi^2-\eta^2)^{2\lambda/\beta} \Phi_3 (2^\lambda/\beta, 2^\lambda/\beta; -\sqrt{\xi^2-\eta^2}; 2\frac{\lambda}{\beta}\sqrt{\xi^2-\eta^2})d\eta d\xi]$$

since the original of $\tilde{g}(\sqrt{s+1})$ is: (see [3]).

$$\frac{1}{2\sqrt{\eta}\ t^{3/2}} [\int_0^\infty \xi e^{-\xi^2/2t} f(\xi)d\xi - \int_0^\infty \xi e^{-\xi^2/4t} \int_0^\infty J_1(\eta) f(\sqrt{\xi^2-\eta^2})d\eta d\xi]$$

and the original of $\tilde{h}(s) = \frac{1}{s^b(1\frac{c}{s})a} \exp(\frac{d}{s})$ is :

$$h(t) = \frac{t^{b-1}}{\Gamma(b)} \Phi_3(a,b;ct,dt)$$

where Φ_3 is a confluent hypergeometric function of two variables (see [10]).

e) CASE $\delta = 3$

We get $\tilde{f}$ after a straight forward integration:

$$f^{\sim}(s) = \frac{e^{-3\lambda/2\beta}}{(1+s)\lambda/\beta} \exp\{\lambda/\beta\ (\frac{1}{1+s} + \frac{1}{2(1+s)^2})\}$$

but the original of

$$\frac{1}{s^{\alpha}} \exp\left(\frac{\gamma}{s}\right) \quad \text{is} \quad \left(\frac{t}{\gamma}\right)^{(\alpha-1)/2} I_{\alpha-1}(2\sqrt{\gamma t})$$

and the original of

$$\frac{1}{s} e^{\frac{\gamma}{2s^2}} \quad \text{is} \quad (\text{see [9]}) \; {}_0F_2\left(1, 1/2; \frac{\gamma t^2}{8}\right)$$

f(t) is then the convolution of these two distributions multiplied by e^{-t}.

$$f(t) = e^{-3\lambda/2\beta}\, e^{-t} \int_0^t {}_0F_2\left(1, \frac{1}{2}; \frac{\lambda}{\beta}\, \frac{(t-\xi)^2}{8}\right) \left(\frac{\beta\xi}{\lambda}\right)^{\lambda/2\beta^{-1}} .$$

$$\cdot \; I_{\lambda/\beta-2} \; (2\sqrt{\lambda\xi/\beta})\; d\xi$$

$$\text{for } t \geqslant 0.$$

Where ${}_pF_q$ is a hypergeometrical function.

3) <u>BOTH SIGNS PULSES.</u>

If the pulses delivered have both signs with equal probability, then the Laplace transform of the probability density of the stationary signal V_∞ is given by [6]

$$f^{\sim}(s) = \exp\left\{\frac{\lambda}{2\beta} \int_0^s \frac{h^{\sim}(\xi) + h^{\sim}(-\xi) - 2}{\xi}\, d\xi\right\}.$$

Suppose h is gamma distributed

a) CASE $\delta = 1$ see [1].
b) CASE $\delta = 2$.

We get for $f^{\sim}$ after a straight forward integration:

$$f^{\sim}(s) = \frac{e^{-\lambda/2\beta}}{(1-s^2)^{\lambda/2\beta}} \exp\left(\frac{\lambda}{2\beta(1-s^2)}\right)$$

The corresponding probability density for V_∞ is: [3]

$$f(t) = \frac{e^{-\lambda/2\beta}|t|^{(\lambda/\beta-1)/2}}{2^{(\lambda/\beta-1)/2}\sqrt{\pi}} \sum_{n=o}^{\infty} \frac{(\lambda/\beta)^n|t|^n}{n!\,\Gamma(\lambda/2\beta+n)2^{2n}} K_{\frac{\lambda/\beta-1}{2}+n}(|t|) \quad -\infty<t<+\infty$$

Where K_α is the modified Bessel function of second kind, i.e. $|V_\infty|$ is then distributed according to the difference of 2 non central χ^2 with λ/β degrees of freedom and a non-centrality parameter equal to λ/β. Another example of both signs pulses can be found in ref. [7].

CASE $\delta = 0.5$.

$$f^{\sim}(s) = \frac{4^{\lambda/\beta}}{((1+\sqrt{1+s})(1+\sqrt{1-s}))^{2\lambda/\beta}}$$

but

$${}_2F_1\,(a, a-1/2;\, 2a;\, s) = \left(\frac{1+\sqrt{1-s}}{2}\right)^{1-2a}$$

where ${}_2F_1$ is the hypergeometric function

$$a = \lambda/\beta + 1/2$$

The product is expanded into (see (12)).

$$\sum_{n=o}^{\infty} \frac{(a)_n\,(a-1/2)_n\,(a)_n\,(a+1/2)_n}{(2a)_n\,(2a)_{2n}\,n!} (-s^2)^n\, {}_2F_1\,(a+n,\, a+n-1/2; 2(a+n); s^2)$$

by the Legendre duplication formula we get:

$$\frac{(a)_n\,(a+1/2)_n}{(2a)_{2n}} = \frac{1}{4}\,n$$

using the Meijer G function (see (12)) we get:

$$G^{2,1}_{2,2}\left(-\frac{1}{s^2}\,\middle|\,{1,\ 2(a+n) \atop a+n,\ a+n-1/2}\right) = \frac{\Gamma(a+n)\ \Gamma(a+n-1/2)}{\Gamma(2(a+n))}$$

$${}_2F_1\ (a+n,\ a+n-1/2;\ 2(a+n);\ s^2)$$

using an identity of the Meijer functions, we get:

$$\tilde{f}(s) = \frac{2^{2a-1}(a-1/2)}{\sqrt{\pi}} \sum_{n=o}^{\infty} \frac{\Gamma(2(a+n))}{\Gamma(2a+n)n!} \frac{1}{4}\, n\ G^{2,1}_{2,2}\left(-\frac{1}{s^2}\,\middle|\,{1-n,\ 2a+n \atop a,\ a-1/2}\right)$$

But the original of

$$\frac{i}{s}\ G^{2,3}_{4,4}\left(-\frac{1}{s^2}\,\middle|\,{o,1/2,1/2-n,2a+n-1/2 \atop a-1/2,\ a-1,\ o,\ 1/2}\right)$$

is known to: (see (13)).

$$\frac{i\ \sqrt{\pi}}{2}\ G^{2,1}_{2,4}\left(-\frac{t^2}{4}\,\middle|\,{1/2-n,\ 2a+n-1/2 \atop a-1/2,\ a-1,o,1/2}\right)$$

Then we get:

$$f(t) = \frac{2^{2a-1}(a-1/2)}{|t|} \sum_{n=o}^{\infty} \frac{\Gamma(2(a+n))}{\Gamma(2a+n)n} \frac{1}{4}\, n\ G^{2,1}_{2,4}\left(-\frac{t^2}{4}\,\middle|\,{1-n,\ 2a+n \atop a-1/2,a,1,1/2}\right)$$

$G^{2,1}_{2,4}$ which is a linear combination of generalized hypergeometric functions can only be easily calculated for special cases, for instance: $\lambda/\beta = 1,\ 1/2$.

In section 2 and 3, the analysis was restricted to the values of the parameter δ for which a closed form for the density has been found.

4) GENERAL FORMULA.

Differentiating the Laplace transform of the stationary density of probability, we get (see (6) and (18))

$$\frac{\partial \tilde{f}(s)}{\partial s} = \frac{\lambda}{\beta} \frac{\tilde{h}(s)-1}{s} \exp \{\frac{\lambda}{\beta} \int_o^s \frac{\tilde{h}(\xi)-1}{\xi} d\xi\}$$

Then we get the mean:

$$\mu = - (\frac{\partial \tilde{f}(s)}{\partial s})_{s=o} = \frac{\lambda}{\beta} \lim_{s \to o} \frac{1-\tilde{h}(s)}{s} = \frac{\lambda}{\beta} \nu$$

since the integral cancels for $s \to o$, $\tilde{h}$ being the Laplace transform of a density probability and ν the mean of this density.

If μ and ν are known from samples of observations we get the following estimate:

$$\hat{\lambda} = \frac{\hat{\mu}}{\hat{\nu}} \beta$$

The accuracy of this formula is subject to the errors on the measurement and to the variances on μ and ν.

IV) Conclusion.

This experiment is not only of academic interest since it is possible to try, using a circuit with variable time constant, to measure the parameter λ of the Poisson distribution corresponding to an unknown radioactive source, where N is the number of particles emitted during the time t.

$$\Pr (N=n) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n =, 0,1,...$$

using $\beta = \frac{1}{RC}$, the time constant and refering to the previous analysis.

This method doesnt pretend to replace a crude counting during a given period of time.

But this method could be of interest when the data acquisition system is not sufficiently rapid to count the particles emitted by a source with a high rate, since in that case the records are only made after a time interval $(0,n/\lambda)$ (where $n\to\infty$), in view to calculate the mean of the distribution of $V_{n/\lambda}$ and to compute λ from β.

BIBLIOGRAPHIC NOTE.

Additional results concerning stochastic models giving rise to sum of products of random variables can be found in references 14, 15, 16, 17.

A reference text book on the physical generators of random numbers is given in ref. (19).

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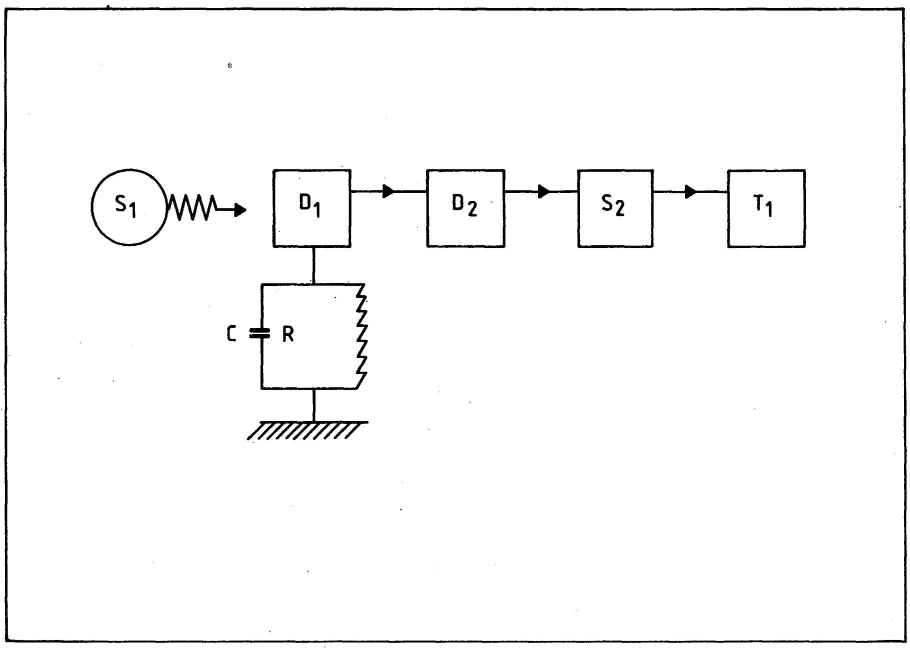

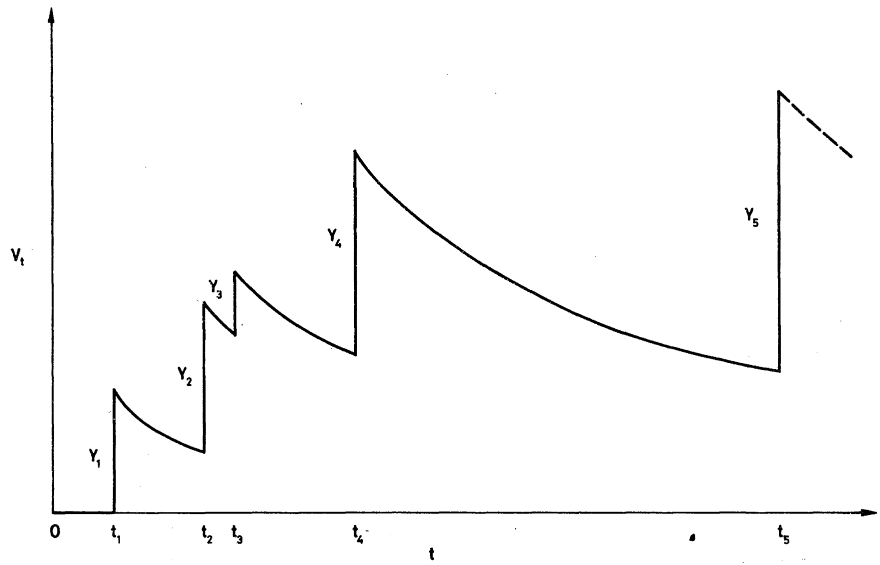

FIG.1

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