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DECOMPOSITION OF TWO PARAMETER MARTINGALES

D. Nualart

ABSTRACT

In this paper we exhibit some decompositions in orthogonal stochastic integrals of two-parameter square integrable martingales adapted to a Brownian sheet which generalize the representation theorem of E. Wong and M. Zakai ([6]). Concretely, a development in a series of multiple stochastic integrals is obtained for such martingales. These results are applied to the characterization of martingales of path independent variation.

0. Introduction.

Stochastic integration with respect to two-parameter martingales was first developed by R. Cairoli and J.B. Walsh [1]. It was observed that certain types of integrals are only defined for strong martingales, which in the Brownian case can be written as $M_z=\int_{R_z} \phi dW$. These martingales have path independent variation, and it was conjectured that path independent variation is another characterization of strong martingales.

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The Wong-Zakai representation ([6]) of a two-parameter square integrable martingale adapted to a Brownian motion states that $M_z=\int_{R_z}\phi dW+\int_{R_z}\int_{R_z}\psi dWdW$. In [1] it is shown that if M_z has path independent variation and $\psi(z_1,z_2)$ depends only on $z_1 v z_2$, then $\psi=0$. This result is extended in [5], assuming weaker conditions on the process ψ.

Using the notion of stable subspace of two-parameter square integrable martingales introduced by M. Zakai ([9]), we generalize the Wong-Zakai representation, and a development in a series of multiple orthogonal stochastic integrals is obtained.

This decomposition is used to show that under some regularity conditions a path independent variation martingale is strong.

We remark that in [9] another class of two-parameter martingales is introduced (martingales of orthogonal increments) which characterize the strong martingales in the Brownian case.

1. Subspaces of two-parameter square integrable martingales

Let $W=\{W_z,\ z\in T\}$, $T=[0,1]^2$, be a two-parameter Wiener process in a completed probability space $(\Omega,\mathcal{F},P)$, that is, a Gaussian separable process with zero mean and covariance function given by $E[W_{s_1t_1}\cdot W_{s_2t_2}] = (s_1\wedge s_2)\cdot(t_1\wedge t_2)$.

If $(s_1,t_1),(s_2,t_2)$ are points of T, we will consider the usual order $(s_1,t_1)\leqslant(s_2,t_2)$ if and only if $s_1\leqslant s_2$ and $t_1\leqslant t_2$. $(s_1,t_1)<(s_2,t_2)$ means that $s_1<s_2$ and $t_1<t_2$, and we will write $(s_1,t_1)\wedge(s_2,t_2)$ if $s_1\leqslant s_2$ and $t_1\geqslant t_2$. If $z_1<z_2$, $(z_1,z_2]$ denotes the rectangle $\{z\in T/z_1<z\leqslant z_2\}$.

Denote by $\{\mathcal{F}_z,\ z\in T\}$ the increasing family of σ-fields generated by W and the null sets of $\mathcal{F}$. For each $(s,t)\in T$ we will consider the families

$$F_{st}^{1} = \bigvee_{\tau\epsilon[0,1]} F_{s\tau} \quad \text{and} \quad F_{st}^{2} = \bigvee_{\sigma\epsilon[0,1]} F_{\sigma t}.$$

Let E_o be the set $\{(s,t)\epsilon T/s=o \text{ or } t=o\}$.

Let $M=\{M_z, z\epsilon T\}$ be a F_z-adapted, integrable process null on E_o, and for each $z_1<z_2$, $z_1=(s_1,t_1)$, $z_2=(s_2,t_2)$ we put

$$M(z_1,z_2]= M_{z_2} - M_{(s_1,t_2)} - M_{(s_2,t_1)} + M_{z_1}.$$

We recall the following definitions:

1. M_z is a <u>martingale</u> if $E\{M_{z_2}/F_{z_1}\}=M_{z_1}$, for all $z_1\leqslant z_2$.
2. M_z is a <u>strong martingale</u> if $E\{M(z_1 z_2]/F_{z_1}^1 \vee F_{z_1}^2\}= 0$, for all $z_1\leqslant z_2$.
3. M_z is a <u>weak martingale</u> if $E\{M(z_1,z_2]/F_{z_1}\}= 0$, for all $z_1\leqslant z_2$.

Denote by m^2 the space of all square integrable martingales (we identify as usual two versions of the same process) which is a Hilbert space isometric to $L^2(\Omega,F,P)$. Let m_s^2 be the closed subspace of m^2 formed by the strong martingales.

The definition and properties of stable subspaces of m^2 is analogous to the one parameter case, using the notion of stopping set instead of stopping times (see [9]).

<u>Definition 1.1.</u> A <u>simple stopping set</u> $D(\omega)$ is a map from Ω to the subsets of T, of the form

$$D(\omega) = \bigcup_{\{(i,j)/\omega\epsilon A_{ij}\}} (z_{ij}, z_{i+1,j+1}],$$

where $(z_{ij}, z_{i+1,j+1}]$ is a partition of T (these rectangles will be closed if $z_{ij}\epsilon E_o$), $A_{ij}\epsilon F_{z_{ij}}$, and for all $\omega\epsilon\Omega$, $z\epsilon D(\omega)$ implies

$R_z \subset D(\omega)$.

If $M \in m^2$ and D is a simple stopping set we will write $M(D) = \sum_{ij} M(z_{ij}, z_{i+1,j+1}] 1_{A_{ij}}$.

Definition 1.2. A closed linear subspace $\mathcal{H}$ of m^2 is said to be stable if $M \in \mathcal{H}$ implies $\{M(R_z \cap D),\ z \in T\} \in \mathcal{H}$ for all simple stopping sets D.

Two martingales $M, N \in m^2$ are said to be strongly orthogonal if $M_z N_z$ is a weak martingale, and this implies that M and N are orthogonal in m^2. It can be proved (see [9]) that if $\mathcal{H}$ is a stable subspace of m^2, $\mathcal{H}$ is also stable and $\mathcal{H}$ and $\mathcal{H}^{\perp}$ are strongly orthogonal.

Let L^2_w be the class of all F_z-adapted and measurable processes $\phi = \{\phi_z,\ z \in T\}$ such that

$$\int_T E\{\phi(z)^2\}\, dz < \infty.$$

Let L^2_{ww} be the class of all processes

$$\psi = \{\psi(z,z'),\ z,z' \in T\} \quad \text{satisfying:}$$

(i) $\psi(z,z';\omega)$ is measurable and $F_{z \vee z'}$-adapted,

(ii) $\psi(z,z') = 0$ unless $z \wedge z'$,

(iii) $\iint_{T \times T} E\{\psi(z,z')^2\}\, dz\, dz' < \infty$.

The representation theorem of E. Wong and M. Zakai ([6]) states that for all $M \in m^2$ there exists two unique processes $\phi \in L^2_w$, $\psi \in L^2_{ww}$ such that

$$M_{st} = \int_{R_{st}} \phi(z)\, dW_z + \iint_{R_{st} \times R_{st}} \psi(z,z')\, dW_z dW_{z'}. \qquad (1.1)$$

Denote by $m^2_w \subset m^2$ the closed subspace of all martingales of the form.

$$M_{st} = \iint_{R_{st}\times R_{st}} \psi(z,z')\, dW_z dW_{z'}\ , \quad \psi\in L^2_{ww}.$$

With this notation, the theorem of E. Wong and M. Zakai can be expressed as follows.

Proposition 1.1. m^2_s and m^2_w are orthogonal stable subspaces of m^2 and $m^2=m^2_s \oplus m^2_w$. The decomposition given by this direct sum coincides with (1.1).

Proof: The only thing to prove is stability, and it is a consequence of proposition 5.1 of [7].

Let $\hat{L}^2_w$ be the class of all F_z-adapted and measurable processes $\emptyset = \{\emptyset_z,\ z\in T\}$ such that $\int_T E\ \{\emptyset(s,t)^2\}\ st\ ds\ dt<\infty$.

Denote by $m^2_j \subset m^2$ the closed subspace of all martingales of the form

$$M_{st} = \iint_{R_{st}\times R_{st}} \emptyset\ (zvz')\ d\ W_z\ d\ W_{z'},\ \emptyset\in L^2_w.$$

From proposition 5.1 of [7] we know that m^2_j is stable; denote by m^2_k its orthogonal complement in m^2_w (which is also stable).

Proposition 1.2. For each martingale $M\in m^2$, the decomposition given by the direct sum $m^2 = m^2_s \oplus m^2_j \oplus m^2_k$ is

$$M_{st} = \int_{R_{st}} \emptyset(z)dW_z + \iint_{R_{st}\times R_{st}} \emptyset_1(zvz')dW_zdW_{z'} +$$

$$+ \iint_{R_{st}\times R_{st}} \psi(z,z')dW_zdW_{z'}\ , \qquad (1.2)$$

where $\emptyset\in L^2_w$, $\emptyset_1\in L^2_w$, $\psi\in L^2_{ww}$ and $\int_{R_{st}} \Psi(x,t;s,y)dxdy = 0$ a.s., for all $(s,t)\in T$ except on a set of Lebesgue measure zero.

<u>Proof:</u> We have only to prove that a martingale $\iint_{R_{st}\times R_{st}} \psi(z,z')\,dW_z dW_{z'}$ belongs to m_k^2 if and only if the process $\psi \in L^2_{ww}$ verifies

$$\int_{R_{st}} \psi(x,t;s,y)\,dxdy = 0, \text{ for all } (s,t)\in T \text{ and } \omega\in\Omega, \text{ a.e.}$$

Indeed, $\alpha(s,t) = \int_{R_{st}} (x,t;s,y)\,dydx$ verifies

$$\int_T E\{\alpha(s,t)\cdot\ \beta(s,t)\}\,dsdt = 0,$$

for any bounded measurable process β.

In particular, let $f(u,z)$: $R \times T \to R$ be a function with continuous partial derivatives f'_u, f''_u, f'''_u, satisfying

$$\frac{\partial f}{\partial s} + \frac{1}{2}\,.t.\,\frac{\partial^2 f}{\partial u^2} = 0 \quad , \quad \frac{\partial f}{\partial t} + \frac{1}{2}.s.\,\frac{\partial^2 f}{\partial u^2} = 0, \qquad (1.3)$$

and such that $f'_u(W_z,z)\in L^2_w$ and $f''_u(W_z,z)\in \hat{L}^2_w$.

Then we know (see [8]) that the process

$$X_{st} = f(W_{st},s,t)-f(0,0,t)-f(0,s,0)+f(0,0,0)$$

belongs to $m_s^2 \oplus m_j^2$, and

$$X_{st} = \int_{R_{st}} f'_u(W_z,z)\,dW_z + \iint_{R_{st}\times R_{st}} f''_u(W_{zvz'},zvz')\,dW_z dW_{z'}.$$

In the next section we will generalize this kind of orthogonal decompositions.

2. Decomposition theorem.

Denote by L^2_n, $n \geqslant 1$, the class of processes $\{\psi(z,z_1,\dots,z_n);\ z,z_1,\dots,z_n\in T\}$ such that

(i) $\psi(z,z_1,\dots,z_n;\omega)$ is measurable and F_z-adapted.

(ii) $\psi(z,z_1,\dots,z_n)=0$ unless $x\leqslant x_n\leqslant x_{n-1}\dots\leqslant x_1$ and $y_1\vee\dots\vee y_n\leqslant y$, where $z=(x,y)$ and $z_i=(x_i,y_i)$, $i=1,\dots,n$.

(iii) $\int_{T^{n+1}} E\{\psi(z,z_1,\dots,z_n)^2\}dz\, dz_1\dots dz_n<\infty$.

For each process $\psi\in L_n^2$ we can define the following multiple stochastic integral,

$$M_{st}=\int_{(R_{st})^{n+1}}\psi(z,z_1,\dots,z_n)dW_z dW_{z_1}\dots dW_{z_n}, \quad (2.1)$$

where $(s,t)\in T$.

The process $\{M_z, z\in T\}$ is a martingale of m^2 such that

$$E\{M_{st}^2\}=\int_{(R_{st})^{n+1}} E\{\psi(z,z_1,\dots,z_n)^2\}\, dz\, dz_1\dots dz_n.$$

Let $\mathcal{H}_n\subset m^2$, $n\geqslant 1$, be the closed subspace of all martingales of the form (2.1) with $\psi\in L_n^2$. These subspaces are no longer stable but they are mutually orthogonal and strongly orthogonal to m_s^2 because $\mathcal{H}_n\subset m_w^2$ for all $n\geqslant 1$.

Theorem 2.1.

$$m^2=m_s^2\oplus(\oplus_{n\geqslant 1}\mathcal{H}_n).$$

Proof: We have only to prove that $m_w^2=\oplus_{n\geqslant 1}\mathcal{H}_n$. Fix a martingale of m_w^2,

$$M_{st}=\iint_{R_{st}\times R_{st}}\psi(z,z')\, dW_z dW_{z'},\ \text{with } \psi\in L_{ww}^2,$$

and define

$$\psi_1(z,z_1) = E\{\psi(z,z_1)/F_z\}, \text{ for all } z,z_1 \in T.$$

Taking a measurable version of this process, we obtain $\psi_1 \in L_1^2$. Next we make the following decomposition

$$\psi(z,z_1)=\psi_1(z,z_1)+[\psi(z,z_1)-\psi_1(z,z_1)]=\psi_1(z,z_1)+\int_{R_{x_1y}-R_z} \alpha_2(z,z_1,z_2)\,dW_{z_2}$$

where $\{\alpha_2\ (z,z_1,z_2);\ z,z_1,z_2 \in T\}$ verifies

(i') $\alpha_2(z,z_1,z_2;\omega)$ is measurable and $F_{z \vee z_2}$-adapted,

(ii) $\alpha_2(z,z_1,z_2) = 0$ unless $x \leq x_2 \leq x_1$ and $y_1 \vee y_2 \leq y$,

(iii) $\int_{T^3} E\{\alpha_2(z,z_1,z_2)^2\}\,dz\,dz_1\,dz_2 < \infty$.

The existence and properties of this process α_2 would follow from a procedure analogous to the one used by R. Cairoli and J.B. Walsh in [2].

Then we have

$$M_{st} = \iint_{R_{st}\times R_{st}} \psi_1(z,z_1)\,dW_z dW_{z_1} + \int_{(R_{st})^3} \alpha_2(z,z_1,z_2)\,dW_z dW_{z_1} dW_{z_2},$$

where the first integral belongs to $\mathcal{H}_1$.

Now we repeat this decomposition successively; that means, for instance,

$$\psi_2(z,z_1,z_2) = E\{\alpha_2(z,z_1,z_2)/F_z\} \in L_2^2, \quad \text{and}$$

$$\alpha_2(z,z_1,z_2) = \psi_2(z,z_1,z_2)+\int_{R_{x_2y}-R_z} \alpha_3(z,z_1,z_2,z_3)\,dW_{z_3},$$

and we obtain in general,

$$M_{st} = \int_{(R_{st})^2} \psi_1(z,z_1)dW_z dW_{z_1} + \ldots + \int_{(R_{st})^{n+1}} \psi_n(z,z_1,\ldots,z_n)dW_z dW_{z_1}\ldots dW_{z_n} +$$

$$+ \int_{(R_{st})^{n+2}} \alpha_{n+1}(z,z_1,\ldots,z_{n+1})\, dW_z dW_{z_1}\ldots dW_{z_{n+1}} . \qquad (2.2)$$

Observe that the first n integrals on the right member of (2.2) belong respectively to $\mathcal{H}_1,\ldots,\mathcal{H}_n$.

Denote by $L^2_{(n)}$ the class of processes $\{\alpha(z,z_1,\ldots,z_n);\ z_1,\ldots,z_n,z\epsilon T\}$ verifying properties (ii) and (iii) in the definition of L^2_n, and also satisfying

(i') $\alpha(z,z_1,\ldots,z_n;\omega)$ is measurable and F_{zvz_n}-adapted.

Let $\mathcal{H}_{(n)}$ be the closed subspace of m^2, of the martingales

$$M_{st} = \int_{(R_{st})^{n+1}} \alpha(z,z_1,\ldots,z_n)dW_z dW_{z_1} \ldots dW_{z_n} ,$$

with $\alpha \epsilon L^2_{(n)}$.

With this notation, $\alpha_n \epsilon L^2_{(n)}$ for all $n \geqslant 1$ and we have obtained the orthogonal decomposition

$$m^2_w = \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n \oplus \mathcal{H}_{(n+1)} .$$

For each $n \geqslant 1$, denote by p_n and $p_{(n)}$ the projections of m^2_w on $\mathcal{H}_n$ and $\mathcal{H}_{(n)}$, respectively. Then, we want to prove that

$$\lim_n p_{(n)}(M) = 0$$

The sequence

$$\| p_{(n)}(M)\|^2 = \int_{T^{n+1}} E\{\alpha_n(z,z_1,\ldots,z_n)^2\}dzdz_1\ldots dz_n$$

is decreasing because of the orthogonality of the decomposition (2.3).

Let T^N be the set of all sequences $\xi = (z, z_1, \dots, z_n \dots)$ of points of T provided with the product Lebesgue measure. We can consider the sequence of funcions $\varphi_n(\xi) = E\{\alpha_n(z, z_1, \dots, z_n)^2\}$ defined on T^N. These functions verify

$$\int_{T^N} \varphi_n(\xi)\, d\xi = \| p_{(n)}(M) \|^2 .$$

For each n, $\varphi_n(\xi)$ is zero except in the set .

$$A_n = \{\xi \in T^N / x \leqslant x_n \leqslant x_{n-1} \dots \leqslant x_1 \text{ and } y_1 \vee \dots \vee y_n \leqslant y\}.$$

Let $A = \bigcap_{n=1}^{\infty} A_n$. The sets A_n form a decreasing sequence, and the product measure of A in T^N is zero because

$$A \subset \{\xi \in T^N / x_1 \geqslant x_2 \geqslant \dots \geqslant x_n \geqslant \dots\}.$$

Therefore, the sequence $\varphi_n(\xi)$ converges to zero almost everywhere.

Moreover from

$$\alpha_n(z, z_1, \dots, z_n) = \psi_n(z, z_1, \dots, z_n) + \int_{R_{x_n y} - R_z} \alpha_{n+1}(z, z_1, \dots, z_{n+1})\, dW_{z_{n+1}}$$

we deduce that

$$\varphi_n(\xi) = E\{\psi_n(z, z_1, \dots, z_n)^2\} + \int_T \varphi_{n+1}(\xi)\, dz_{n+1}.$$

Thus, $\{\varphi_n(\xi), n \geqslant 1\}$ is a positive supermartingale and, therefore,

$$\lim_n \int_{T^N} \varphi_n(\xi)\, d\xi = 0.$$

In particular, let $f(u,z): R \times T \to R$ be a function with continuous partial derivatives with respect to u of all orders, satisfying (1.3) and such that $f'_u(W_z,z)\in L^2_w$, $f''_u(W_z,z)\in L^2_2$ and $f^{(n)}_u(W_{z\wedge z_n},z\wedge z_n)\in L^2_n$ for all $n\geqslant 3$. Suppose also

$$f^{(n)}_u(0,0,y) = f^{(n)}_u(0,x,0) = 0 \text{ for all } n\geqslant 0.$$

Then, the martingale $M_{st}=f(W_{st},s,t)$ has the following decomposition

$$M_{st}=\int_{R_{st}} f'_u(W_z,z)\,dW_z + \int_{(R_{st})^2} f''_u(W_z,z)\,dW_z dW_{z_1} +$$

$$+\sum_{n\geqslant 3}\int_{(R_{st})^{n+1}} f^{(n)}_u(W_{z\wedge z_n},z\wedge z_n)\; dW_z\, dW_{z_1}\ldots dW_{z_n}.$$

Remark. If we fix the points z and z_1, the infinite expansion for $\psi(z,z_1)$ obtained in theorem 2.1 can be deduced by considering the development of $\psi(z,z_1)$ into multiple Wiener integrals (see Itô's theorem 4.2 of [3], which holds for a two-parameter Brownian motion). These multiple integrals can be represented as iterated stochastic integrals of $\mathcal{F}^1_{st}$-adapted processes (this is the analog of Itô's theorem 5.1 of [3] to the two-parameter case). Using this fact we could evaluate the difference $\psi(z,z_1) - E\{\psi(z,z_1)/\mathcal{F}_z\}$ and get the desired expansion.

3. Martingales of Path Independent Variation.

Let γ the set of all continuous increasing curves on T starting from (0,0). A martingale $M\in m^2$ is said to be of path independent variation if the quadratic variation of M, as a one parameter martingale, along every curve of γ depends only on the end point of the path.

It is easy to see that M is of path independen variation if and only if there exists a unique (excepting modifications) process $\{A_z, z\in T\}$ continuous and increasing on each curve of Υ, such that $A_{(0,0)} = 0$ and $M_z^2 - A_z$ is a martingale.

We know (see [1]) that each strong martingale has path independent variation. The reciprocal of this result is not true as it has been proved in [4]. That means, m_s^2 is a proper subset of the class of path independent variation martingales, which is a closed subset of m^2.

The object of this section is to use the preceding results on martingales decomposition in order to prove this reciprocal in some particular situations.

Let $M\in m^2$ such that

$$M_{st} = \int_{R_{st}} \emptyset(z)\, dW_z + \iint_{R_{st}\times R_{st}} \Psi(z,z')\, dW_z dW_{z'}\,, \tag{3.1}$$

where $\emptyset\in L_w^2$ and $\Psi\in L_{ww}^2$.

In [5] it is shown that if M is of path independent variation, then

$$\int_0^{x'} \Psi(z,z')\, [\emptyset(z) + \int_{R_{x'y}-R_z} \Psi(z,z_1)\, dW_{z_1}]\, dx = 0 \tag{3.2}$$

for all $z' = (x',y')\in T$, $y\in[y',1]$, almost everywhere.

Theorem 2.1 of [5] can be expressed as it follows

<u>Proposition 3.1.</u> If $M\in\mathcal{H}_1$ and M is of path independent variation, then M = 0.

This proposition can be extended to martingales with a finite deterministic integral representation.

<u>Proposition 3.2.</u> Let $M\in m_s^2 \oplus\mathcal{H}_1 \oplus\ldots\oplus \mathcal{H}_n$ be a martingale of path independent variation. If the functions $\emptyset(z)$, $\Psi_1(z,z_1),\ldots,\Psi_n(z,z_1,\ldots,z_n)$ are deterministic, then M is a strong martingale.

<u>Proof:</u> We proceed by induction on n.

For n=1, taking expectations in equation (3.2) and using the fact that $\emptyset(z)$ and $\psi_1(z,z_1)$ are deterministic we obtain

$$\int_0^{x'} \psi_1 \ (z,z') \ \emptyset \ (z) \ dx = 0.$$

Thus,

$$\int_0^{x'} \psi_1(z,z')[\int_{R_{x'y}-R_z} \psi_1(z,z_1) \ dW_{z_1}] \, dx = 0.$$

Now we can commute the integrals because ψ_1 is deterministic, and we obtain

$$\int_0^1 \psi_1(z,z') \ \psi_1(z,z_1) \ dx = 0,$$

for all $y\epsilon[0,1]$, $z',z_1\epsilon T$, $\omega\epsilon\Omega$, almost everywhere with respect to the product Lebesgue measure times the probability P.

Integrating with respect to z' and z_1, we have for any Borel subset B of T,

$$\int_B \psi_1(z,z') \ dz' = 0,$$

for all $z\epsilon T$, $\omega\epsilon\Omega$, a.e., and this implies $\psi_1 = 0$. Therefore, $M\epsilon m_s^2$, and the proposition holds for n=1.

Suppose the proposition proved up to n - 1.

From equation (3.2) we obtain

$$\int_0^{x'} [\sum_{i=1}^{n} \int_{T^{i-1}} \psi_i(z,z',z_1,\ldots,z_{i-1}) \ dW_{z_1} \ldots \ dW_{z_{i-1}}]\emptyset \ (z) \ dx \ +$$

$$+ \int_0^{x'} [\sum_{i=1}^{n} \int_{T^{i-1}} \psi_i(z,z',z_1,\ldots,z_{i-1}) \ dW_{z_1} \ldots dW_{z_{i-1}}] \ . \tag{3.3}$$

$$.[\sum_{j=1}^{n} \int_{T^j} \psi_j(z,\varphi_1,\varphi_2,\ldots,\varphi_j) \ dW_{\varphi_1} \ldots \ dW_{\varphi_j}] \quad dx = 0.$$

Taking expectations recursively in (3.3) we should arrive to an equation of the form

$$\int_0^{x'} \psi_n(z,z',z_1,\dots,z_{n-1})\psi_n(z,\varphi_1,\dots,\varphi_n)dx = 0 \qquad (3.4)$$

for all $(z',z_1,\dots,z_{n-1}) \in T^n$, $(\varphi_1,\dots,\varphi_n) \in T^n$, $y \in [0,1]$, and $\omega \in \Omega$, almost everywhere.

To simplify we are going to deduce equation (3.4) only for n=2. In this case the stochastic differential calculus applied to the product of multiple stochastic integrals of (3.3) give rise to

$$\int_0^{x'} [\psi_1(z,z') + \int_T \psi_2 \ (z,z',z_1) \ dW_{z_1}] \ \emptyset \ (z) \ dx \quad +$$

$$+ \int_0^{x'} [\int_{T^2} \psi_1(z,z') \ \psi_2(z,\varphi_1,\varphi_2) \ dW_{\varphi_1} \ dW_{\varphi_2} \quad +$$

$$+ \ \int_{T^3} \psi_2(z_1,z',z_1) \ \psi_2 \ (z,\varphi_1,\varphi_2) \ dW_{z_1} \ dW_{\varphi_1} \ dW_{\varphi_2} \quad +$$

$$+ \int_{T^2} \psi_2(z,z',\varphi_1) \ \psi_2(z,\varphi_1,\varphi_2) \ d\varphi_1 \ dW_{\varphi_2} \quad +$$

$$+ \int_{T^2} \psi_2(z,z',\varphi_2) \ \psi_2(z,\varphi_1,\varphi_2) \ d\varphi_2 \ dW_{\varphi_1} \ ] \ dx \quad = \ 0.$$

Taking expectations we obtain

$\int_0^{x'} \psi_1(z,z') \ \emptyset \ (z)dx = 0$, and the rest of terms is of the form $\int_T \alpha(\varphi) \ d \ W_\varphi$, so $\alpha(\varphi) = 0$ a.e.

That means

$$\int_0^{x'}\psi_2(z,z',z_1)\ \emptyset(z)dx + \int_T[\int_0^{x'}\psi_1(z,z')\psi_2(z,\varphi_1,\varphi_2)dx]dW_{\varphi_2} +$$

$$+ \int_{(R_{x_1}\bar{y}R_z)^2}[\int_0^{x'}\psi_2(z,z',z_1)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ dW_{\varphi_1}dW_{\varphi_2} +$$

$$+ \int_{(R_{\xi_2}\bar{y}R_z)^2}[\int_0^{x'}\psi_2(z,z',z_1)\psi_2\ (z,\varphi_1,\varphi_2)\ dx]dW_{\varphi_1}\ dW_{z_1} +$$

$$+\int_T[\int_0^{x'}\psi_2\ (z,z',\varphi_1)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ d\varphi_2 +$$

$$+\int_T[\int_0^{x'}\psi_2\ (z,z',\varphi_2)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ d\varphi_1\ ,$$

where $\varphi_2 = (\xi_2,\eta_2)$.

By a similar argument, we have

$$\int_0^{x'}\psi_1(z,x')\psi_2(z,\varphi_1,\varphi_2)\ dx\ +$$

$$+ \{\int_{R_{x_1}\bar{y}R_z}[\int_0^{x'}\psi_2(z,z',z_1)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ dW_{\varphi_1}\}\ .1_{[\xi_2,1]}(x_1)\ +$$

$$+ \{\int_{R_{\xi_1}\bar{y}R_z}[\int_0^{x'}\psi_2(z,z',z_1)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ dW_{z_1}\ +$$

$$+ \int_{R_{x_1}\bar{y}R_z}[\int_0^{x'}\psi_2(z,z',z_1)\psi_2(z,\varphi_1,\varphi_2)\ dx]\ dW_{\varphi_1}\}.1_{[0,\xi_2]}(x_1) = 0.$$

Finally we obtain

$$\int_0^{x'}\psi_2(z,z',z_1)\psi_2(z,\varphi_1,\varphi_2)\ dx\ = 0\ ,$$

for all $(z',z_1)\in T^2$, $(\varphi_1,\varphi_2)\in T^2$, $y\in[0,1]$, $\omega\in\Omega$, a.e., and, therefore, (3.4) holds for n=2.

Now, from (3.4) it is immediate to show that $\psi_n = 0$. Indeed, integrating with respect to $(z',z_1,\dots,z_{n-1})\in T^n$ and $(\varphi_1,\dots,\varphi_n)\in T^n$ we obtain, given a Borel set B of T^n,

$$\int_B \psi_n(z,\alpha)\ d\alpha = 0,$$

for all $z\in T$, $\omega\in\Omega$, a.e. Therefore, $\psi_n = 0$.

Then, $M\in m_s \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1}$ and, by recurrence, $M\in m_s$.

The deterministic property can be replaced by a regularity condition as it is shown in the next proposition.

<u>Proposition 3.3.</u> Let $M\in m_s \oplus \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ be a martingale of path independent variation. If the functions $\psi_1(z,z_1),\dots,\psi_n(z,z_1,\dots,z_n)$ are continuous and have continuous partial derivatives with respect to x_1, for all $\omega\in\Omega$ and $z,z_1,\dots,z_n\in T$, almost everywhere, then M is a strong martingale.

<u>Proof:</u> Like before we proceed by induction on n.

For n=1, consider the equation (3.2).

The term $\int_0^{x'}\psi_1(z,z')\,\phi(z)dx$ is derivable with respect to x', for all $y\in[y',1]$, $z'\in T$ and $\omega\in\Omega$, almost everywhere.

Then, if we write

$$\alpha(y,z',z_1) = \int_0^{x'} \psi_1(z,z')\psi_1(z,z_1)\ dx \text{ and}$$

$$Y(x',\omega) = \int_{R_{x\bar{1}y}R_z} \alpha(y,z',z_1)\ dW_{z_1},$$

we obtain, using the mean value theorem

$$Y(x'+\xi) - Y(x') = \int_{R_{(x'+\xi,y)}-R_z} \xi\cdot\frac{\partial\alpha}{\partial x'}(y,z',z_1)\,dW_{z_1} +$$

$$+ \int_{R_{(x'+\xi)y}-R_{x'y}} \alpha(y,z',z_1)\,dW_{z_1}.$$

So, the limit

$$\lim_{\xi\to 0} \frac{1}{\xi} \int_{R_{(x'+\xi)y}-R_{x'y}} \alpha(y,z',z_1)\,dW_{z_1}, \text{ exists a. e.}$$

and, therefore, $\alpha(y,z',z_1) = 0$. Thus, as in proposition 3.2. this implies $\psi_1 = 0$, and so $M \in m_s$.

Suppose the proposition proved up to n-1 and start with equation (3.3). Using recursively a reasoning analogous to the preceding one we would obtain equation (3.4) and the proof would follow as in proposition (3.2). We omit the details of this process.

From these two propositions we could conclude that non strong path independent variation martingales, which certainly exist (see [4]) cannot have, however, a finite representation of the preceeding form with deterministic or regular integrating processes.

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Departament d'Estadística.
Facultat de Matemàtiques.
Universitat de Barcelona.
Gran Via, 585, Barcelona-7.
SPAIN.