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ON A REPRESENTATION THEOREM OF DE MORGAN ALGEBRAS BY FUZZY SETS

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ABSTRACT

Once the concept of De Morgan Algebra of Fuzzy Sets on a universe X can be defined, we give a necessary and sufficient condition for a De Morgan Algebra to be isomorphic to (represented by) a De Morgan Algebra of Fuzzy Sets.

De Morgan Algebras of Fuzzy Sets.

Let X be a universe of discourse. We denote by $L(X)=(\underset{\sim}{P}(X),\cap,\cup)$ the lattice of fuzzy subsets of X with values on [0,1] being $\cup$ and $\cap$ the usual max and min operations and $\underset{\sim}{P}(X)=[0,1]^X$. It is known that L(X) is a complete, infinitely distributive lattice with maximum $X(X(x)=1$ for any $x\in X)$ and minimum $\emptyset(\emptyset(x)=0$ for any $x\in X)$, and also that boolean elements of L(X) constitute the Boolean Algebra P(X) of crisp subsets of X. We denote by δ_x the singletons of P(X), i.e., $\delta_x:X\to[0,1]$ is defined by $\delta_x(a)=0$ if $a\neq x$, $\delta_x(x)=1$, and by $\underset{\sim}{\alpha}$, $\alpha\in[0,1]$, the constant function $\underset{\sim}{\alpha}(x)=\alpha$ for any $x\in X$. Then we have:

$$A=\bigcup_{x\in X}(\delta_x\cap \underset{\sim}{A(x)})=\bigcup_{x\in X}(\delta_x\cap A),\ \text{for any } A\in\underset{\sim}{P}(X).$$

Definition 1. We say that a sublattice S of L(X) satisfies the Extension Principle (E.P. from now on) if $S \supset P(X)$.

Examples.

Let X be a univers and $\mathbf{J}=\{J \mid \{0,1\} \subset J \subset [0,1]\}$. For any family $\mathcal{J}=\{J_x \in \mathbf{J} \mid x \in X\}$ we define:

$$P_{\mathcal{J}}(X) = \{A \in \underset{\sim}{P}(X) \mid A(x) \in J_x \text{ for any } x \in X\},$$

$$P'_{\mathcal{J}}(X) = \{A \in P_{\mathcal{J}}(X) \mid \{A(x) \mid A(x) \notin \{0,1\}\} \text{is finite}\}.$$

Clearly, both $P_{\mathcal{J}}(X)$ and $P'_{\mathcal{J}}(X)$ are sublattices of L(X) such that satisfy the E.P. It is also easy to verify that $P_{\mathcal{J}}(X)=P'_{\mathcal{J}}(X)$ if and only if X is finite, and $P_{\mathcal{J}}(X)$ is complete if and only if any $J_x \in \mathcal{J}$ is a complete subchain of [0,1].

Proposition 1. A sublattice S of L(X) satisfies the E.P. iff there exists a family $\mathcal{J}=\{J_x \in \mathbf{J} \mid x \in X\}$ such that $P'_{\mathcal{J}}(X) \subset S \subset P_{\mathcal{J}}(X)$.

Proof: If S is a sublattice of L(X) such that satisfies the E.P. we can define, for any $x \in X$, $J_x = \{\alpha \in [0,1] \mid$ there exists $B \in S$ such that $B(x) = \alpha\}$, and we have the family $\mathcal{J}=\{J_x \mid x \in X\}$. We will prove that $P'_{\mathcal{J}}(X) \subset S \subset P_{\mathcal{J}}(X)$.

a) If we denote by $[\emptyset, \delta_x]$ the interval of L(X) defined by $[\emptyset, \delta_x] = \{A \in \underset{\sim}{P}(X) \mid \emptyset \leqslant A \leqslant \delta_x\} = \{A \in \underset{\sim}{P}(X) \mid A(a) = 0 \text{ for any } a \neq x\}$, then $[\emptyset, \delta_x] \cap P_{\mathcal{J}}(X) = [\emptyset, \delta_x] \cap P'_{\mathcal{J}}(X) = [\emptyset, \delta_x] \cap S = \{A \in [\emptyset, \delta_x] \mid A(x) \in J_x\}$. Taken into account the definition of $P_{\mathcal{J}}(X)$ and $P'_{\mathcal{J}}(X)$ it is clear that $[\emptyset, \delta_x] \cap P_{\mathcal{J}}(X) = [\emptyset, \delta_x] \cap P'_{\mathcal{J}}(X) = \{A \in [\emptyset, \delta_x] \mid A(x) \in J_x\}$, and also that $[\emptyset, \delta_x] \cap S \subset \{A \in [\emptyset, \delta_x] \mid A(x) \in J_x\}$. We need to prove that if $A \in [\emptyset, \delta_x]$ and $A(x) \in J_x$, then $A \in S$. Because of the definition of J_x, as $A(x) \in J_x$, there exists a $B \in S$ such that $B(x)=A(x)$. Then $A=\delta_x \cap B$ which proves that $A \in S$ as $\delta_x \in P(X) \subset S$ and $B \in S$.

b) Any $A \in S$ satisfies $A(x) \in J_x$ for any $x \in X$, so $A \in P_{\mathcal{J}}(X)$ and $S \subset P_{\mathcal{J}}(X)$.

c) If $A \in P'_{\mathcal{J}}(X)$ and $X_1 = \{x \in X \mid A(x) = 1\}$, $X_2 = \{x \in X \mid A(x) \notin \{0,1\}\}$, then $A = (\bigcup_{x \in X_1} \delta_x) \cup (\bigcup_{x \in X_2} (\delta_x \cap A))$; but $\bigcup_{x \in X_2} \delta_x \in P(X) \subset S$, X_2 is finite and $\delta_x \cap A \in S$, thus we have $\bigcup_{x \in X_2} (\delta_x \cap A) \in S$. We conclude that $A \in S$ for any $A \in P'_{\mathcal{J}}(X)$ and $P'_{\mathcal{J}}(X) \subset S$.

Reciprocally, if S is a sublattice of L(X) and there exists a family $\mathcal{J}$ such that $P'_{\mathcal{J}}(X) \subset S \subset P_{\mathcal{J}}(X)$, then S satisfies the E.P. $(S \supset P'_{\mathcal{J}}(X) \supset P(X))$.

Definition 2. A De Morgan Algebra $M(X) = (S, \cap, \cup, n)$ is said to be a De Morgan Algebra of Fuzzy Sets on X if S is a sublattice of L(X) that satisfies the E.P.

Examples.

1) Let $X = \{x,y\}$, $J_x = \{1/(n+1) \mid n \in N\} \cup \{0,1\}$ $J_y = \{n/(n+1) \mid n \in N\} \cup \{0,1\}$ and $\mathcal{J} = \{J_x, J_y\}$. We consider the sublattice $P_{\mathcal{J}}(X)$ of L(X). From mappings $n_x : J_x \to J_y$ defined by $n_x(0) = 1$, $n_x(1) = 0$, $n_x(1/(n+1)) = n/(n+1)$ for any $n \in N$, and $n_y : J_y \to J_x$ being n_y the inverse of n_x, a strong negation n on $P_{\mathcal{J}}(X)$ can be defined in the following way:

$$(n(A))(x) = n_y(A(y)) \quad , \quad (n(A))(y) = n_x(A(x)) \text{ for any } A \in P_{\mathcal{J}}(X).$$

It is easy to verify that n is a strong negation and $(P_{\mathcal{J}}(X), \cap, \cup, n)$ is a De Morgan Algebra of Fuzzy Sets.

2) Let $X = \{x,y\}$, $J_x = \{0,1\}$, $J_y = [0, \frac{1}{2}) \cup \{1\}$ and $\mathcal{J} = \{J_x, J_y\}$. We consider the sublattice $P_{\mathcal{J}}(X)$ of L(X). From $n_x : J_x \to J_x$, defined by $n_x(0) = 1$, $n_x(1) = 0$, and $n_y : J_y \to J_y$, defined by $n_y(0) = 1$, $n_y(1) = 0$, $n_y(\alpha) = \frac{1}{2} - \alpha$ for any $\alpha \in (0, \frac{1}{2})$, a strong negation n on $P_{\mathcal{J}}(X)$ can be defined in the following way:

$$(n(A))(a) = n_a(A(a)) \text{ for any } A \in P_{\mathcal{J}}(X) \text{ and any } a \in X.$$

It is easy to verify that n is an strong negation and $(P_J(X), \cap, \cup, n)$ is a De Morgan Algebra of Fuzzy Sets.

Remark. Clearly not every De Morgan Algebra of Fuzzy Sets is a subalgebra of a De Morgan Algebra on L(X) (see [2],[3]). For instance, $P_J(X)$ in example 1 is clearly a subalgebra of a De Morgan Algebra on L(X) but this is not the case in example 2 since n_x, n_y from example 1 are restrictions to J_x, J_y of negation functions on [0,1], whereas n_y from example 2 could never be a restriction of a negation function on [0,1].

A representation theorem.

Given a De Morgan Algebra $A=(A,\wedge,v,n)$ with maximum u and minimum o, we will find the conditions required so that it is isomorphic to a De Morgan Algebra of Fuzzy Sets on a X.

Definition 3. A De Morgan Algebra A is said to be of Fuzzy type if there exists a set X and a De Morgan Algebra of Fuzzy Sets S on X, such that A and S are isomorphic.

Note that $f:(A,\wedge,v,n) \to (S,\cap,\cup,\bar{n})$ is a morphism if it satisfies:

a) $f(a \vee b) = f(a) \cup f(b)$,

b) $f(a \wedge b) = f(a) \cap f(b)$,

c) $f(n(a)) = \bar{n}(f(a))$.

Proposition 2. A is of Fuzzy Type iff there exist a universe X and a sublattice S of L(X) which contains P(X) and such that $(A,\wedge,v)$ and $(S,\cap,\cup)$ are isomorphic when considered as lattices.

Proof: If A is of Fuzzy type the condition clearly holds. Conversely, if there exists X, a sublattice S of L(X) which contains P(X) and a isomorphism $f:(A,\wedge,v) \to (S,\cap,\cup)$, then A is of

Fuzzy type since $\bar{n}:S \to S$ defined by $\bar{n}(A)=f(n(f^{-1}(A)))$, for any $A\in S$, is a strong negation on S and f is an isomorphism between the De Morgan Algebras $(A,\wedge,v,n)$ and $(S,\cap,\cup,\bar{n})$.

<u>Theorem.</u> A De Morgan Algebra $\mathcal{A}=(A,\wedge,v,n)$ is of Fuzzy type if and only if A satisfies the following conditions:

a) The Boolean Algebra B of the Boolean elements of the distributive lattices $(A,v,\wedge)$ is complete and atomic;

b) For every atom x of B there exists $J_x\in\mathcal{J}$ and an isomorphism $\sigma_x:[0,x] \to J_x$;

c) For every pair of atoms x,y of B such that $x \neq y$, $[o,x] \cap [o,y] = \{o\}$;

d) For every $a\in A$, it is $a = \bigvee_{x\in X}(a\wedge x)$, where $X = \{x\in B \mid x$ is an atom of $B\}$.

Proof: If A is of Fuzzy type there exists a De Morgan Algebra of Fuzzy Sets S on a universe X such that $(A,\wedge,v,n)$ is isomorphic to $(S,\cap,\cup,\bar{n})$. It is clear that the Algebra of Boolean elements of S is P(X) which is complete and atomic, that the atoms of P(X) satisfy b) and c) and that for every $A\in S$ is $A=\bigcup_{x\in X}(\delta_x\cap A)$, and, therefore, the same conditions must hold for A.

Conversely, if A satisfies the conditions of the theorem, then since B is complete and atomic $B\simeq P(X)$ being $X=\{x\in B \mid x$ is atom of $B\}$. If we denote by $\mathcal{J}$ the family $\mathcal{J}=\{J_x \mid x\in X\}$, where J_x are the intervals given in b), we will prove that a one-to-one morphism can be stablished between A and $P_{\mathcal{J}}(X)$.

We define $f:(A,\wedge,v) \to (S,\cap,\cup)$ by $f(a)=\bigcup_{x\in X}(\delta_x\cap\sigma_x(a\wedge x))$.

Firstly it is easy to prove that f is a morphism so it is only necessary to see that $(f(a))(x) = \sigma_x(a\wedge x)$ for any $x\in X$ and to take into account that σ_x is an isomorphism.

Secondly f is a one-to-one morphism since if $f(a)=f(b)$, for any $x\in X$ $(f(a))(x)=(f(b))(x)$ that is $\sigma_x(a\wedge x)=\sigma_x(b\wedge x)$ that is

$a\wedge x = b\wedge x$ and so $a = \bigcup_{x\in X}(a\wedge x) = \bigcup_{x\in X}(b\wedge x) = b$.

Lastly $f(A)\supset P(X)$ since $f(B) = P(X)$ because, for every $y\in X$, it is $f(y)=\bigcup_{x\in X}(\delta_x \cap\sigma_x(y\wedge x)) = \delta_y$.

Therefore f is an isomorphism between the lattice $(A,\wedge,v)$ and the sublattice $(f(A),\cap,\cup)$, of $L(X)$, which contains $P(X)$. In accordance with proposition 1 this proves that A is of Fuzzy type.

Remarks. 1) It is easy to see that, the condition b) can be substituted by: b') "For every atom x of B there exists a one-to-one morphism $\sigma_x[0,x] \to [0,1]$". In that case we take $J_x=\sigma_x([0,1])$ and the proof is the same.

2) It is also clear that, in general, condition c) can be replaced by: c') "For every pair of atoms x,y of B such that $x \neq y$, any $\alpha,\beta\in A$ such that $o\leqslant\alpha\leqslant x$, $o\leqslant\beta\leqslant y$ satisfy $\alpha\wedge\beta=o$".

3) Besides, it is easy to prove that, for A to be isomorphic to a De Morgan Algebra on $\underset{\sim}{P}(X)$ or on any complete sublattice containing $P(X)$ the condition d) should be replaced by: d') "A is complete and infinitely distributive lattice". In such a case d) is deduced form d') as follows:

For every $a\in A$, it is $a = u\wedge a = (\bigvee_{x\in X}x)\wedge a = \bigvee_{x\in X}(x\wedge a)$.

Besides, it can be proved that f is a bijection between A and a complete sublattice $S\subset\underset{\sim}{P}(X)$ as f is onto:

for any $A\in S$ there is $a=\bigcup_{x\in X} \sigma_x^{-1}(x\wedge a)$ such that $f(a)=\bigcup_{x\in X}(\delta_x\cap\sigma_x(a\wedge x))$ $=\bigcup_{x\in X}(\delta_x \cap A(x)) = A$.

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