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REPRESENTATIONS OF SYNONYMY AND ANTONYMY BY AUTOMORPHISMS IN FUZZY SET THEORY *

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ABSTRACT

Structures of automorphisms and automorphism groups in fuzzy set theory are studied in detail in view of applications to synonymy and antonymy representations.

1. Introduction.

The aim of this paper is to suggest an algebraic model which may provide an answer to the following Zadeh's question: how could synonyms and antonyms be represented in fuzzy set theory?.

Let us suppose that there is a rule assigning a synonym (or an antonym) to each fuzzy set with a given universe. It is easy to accept a hypothesis that this rule commutes with connectives "and" and "or". For instance, "wealthy and sick" is an antonym to "poor and healthy". It means that the rule in question is actually an automorphism of an algebra of all fuzzy sets. Obviously, there are a lot of possible rules of this kind. Symmetry of synonymity and statements in a colloquial language like "a synonym of

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a synonym is a synonym" or "an antonym of an antonym is a synonym" show that a proper mathematical model should employ a group structure of a set of automorphisms.

The paper studies automorphisms in fuzzy set theory (section 2) and automorphism groups (section 3) with the view of their applications to synonymy representations. Only algebraic aspects of the problem in question are considered in this paper. We leave applications to linguistics for further publications.

2. Automorphisms in fuzzy set theory.

Let X be a finite set. Fuzzy set theory considers the following model. A fuzzy set A with universe X is a mapping $A:X \to [0,1]$. A function $A(x)$ with domain X and range $[0,1]$ is said to be a membership function. Further we will not distinguish between fuzzy sets and their membership functions. The set of all fuzzy sets with universe X is denoted $\underset{\sim}{P}(X)$. Operations of union and intersection are defined pointwise by

$$(A \cup B)(x) = A(x) \vee B(x) = \max\{A(x), B(x)\},$$

$$(A \cap B)(x) = A(x) \wedge B(x) = \min\{A(x), B(x)\}.$$

The set $\underset{\sim}{P}(X)$ is a complete completely distributive lattice with respect to operations $\cup$ and $\cap$ and universal bounds 0 and 1 where $0(x) \equiv 0$ and $1(x) \equiv 1$. Considering this lattice as an abstract algebra we denote $L(X) = \langle P(X); \cup, \cap, 0, 1\rangle$. Actually, $L(X) = X^{[0.1]}$ where $[0,1]$ is regarded as a lattice with respect to max- and min-operations. An operation of negation $^{-}$ is defined as follows in fuzzy set theory

$$\bar{A}(x) = 1 - A(x), \text{ for all } x \in X.$$

The lattice $L(X)$ endowed with a negation operation defined above is a de Morgan algebra $M(X) = \langle P(X); \cup, \cap, {}^{-}, 0, 1 \rangle$ (see [1] for a general definition of de Morgan algebras).

An <u>automorphism</u> of $L(X)$ is a one-to-one and onto mapping $\theta: \underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ such that

$$\theta(A \cup B) = \theta(A) \cup \theta(B),$$
$$\theta(A \cap B) = \theta(A) \cap \theta(B),$$
$$\theta(0) = 0 \text{ and } \theta(1) = 1.$$

We obtain an automorphism of $M(X)$ adding the property

$$\theta(\bar{A}) = \overline{\theta(A)}.$$

In this section all automorphisms of $L(X)$ and $M(X)$ are completely described. We start with a description of automorphisms of $L(X)$, for any automorphism of $M(X)$ is an automorphism of $L(X)$.

Let us denote $P(X)$ a set of all crisp subsets in X, i.e. fuzzy sets with membership functions taken only values 0 and 1. Then $P(X)$ is a Boolean algebra which is a maximal Boolean subalgebra in $L(X)$ (and $M(X)$).

<u>Lemma 2.1.</u> Let θ be an automorphism of $L(X)$. Then the restriction of θ on $P(X)$ is an automorphism of $P(X)$.

<u>Proof.</u> Let A be a crisp set, i.e. $A \in P(X)$. Then

$$A \cup \bar{A} = 1 \text{ and } A \cap \bar{A} = 0$$

imply

$$\theta(A) \cup \theta(\bar{A}) = 1 \text{ and } \theta(A) \cap \theta(\bar{A}) = 0.$$

Hence, $\theta(A)$ is a crisp set and $\theta(\bar{A}) = \overline{\theta(A)}$. ■

<u>Lemma 2.2.</u> Let θ be an automorphism of $P(X)$. There is a permutation $s:X \to X$ such that

$$\theta(A)(x) = A(s(x)) \text{ for any } A \in P(X).$$

<u>Proof.</u> Atoms in $P(X)$ are singletons in X. An image and an inverse image of any atom are atoms again, for θ is an automorphism. Hence, θ defines a permutation on the set X. Note now that any $A \in P(X)$ is a union of atoms. ■

<u>Remark.</u> The group of all automorphisms of $P(X)$ is isomorphic to the symmetric group S_n for $n = |X|$.

In order to describe automorphisms of $L(X)$ we introduce the following families of elements in $L(X)$:

$$\delta_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a, \end{cases} \quad \text{for } a \in X$$

and

$$\sigma_\alpha(x) \equiv \alpha \qquad \text{for } \alpha \in [0,1].$$

Note that $\delta_a(x)$ is an atom in $P(X)$ for any $a \in X$.

Let θ be an automorphism of $L(X)$. We define

$$\theta_x(\alpha) = \theta(\sigma_\alpha)(x) \quad \text{for } x \in X,\ \alpha \in [0,1].$$

<u>Lemma 2.3.</u> θ_x is an automorphism of $[0,1]$ for any $x \in X$.

<u>Proof.</u> We have

$$\theta_x(\alpha,\beta) = \theta(\sigma_{\alpha\wedge\beta})(x) = \theta(\sigma_\alpha \wedge \sigma_\beta)(x) =$$

$$\theta(\sigma_\alpha)(x) \wedge \theta(\sigma_\beta)(x) = \theta_x(\alpha) \wedge \theta_x(\beta).$$

In the same way $\theta_x(\alpha \vee \beta) = \theta_x(\alpha) \vee \theta_x(\beta)$. Finally, $\theta_x(0) = 0$ and $\theta_x(1) = 1$.

Now we have the following

Theorem 2.1. A mapping $\theta: \underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ is an automorphism of the lattice L(X) iff there is a family $\{\theta_x\}$, $x \in X$, of automorphisms of $[0,1]$ and a permutation $s: X \to X$ such that

$$\theta(A)(x) = \theta_x(A(s(x))) \tag{2.1}$$

for any $A \in \underset{\sim}{P}(X)$.

Proof. It is easy to verify that (2.1) defines an automorphism of L(X) for any family $\{\theta_x\}$ and any permutation s.

Conversely, let $A \in \underset{\sim}{P}(X)$. Then we have a decomposition

$$A(x) = \bigcup_{a \in X} \{\delta_a(x) \wedge \sigma_{A(a)}(x)\}.$$

Hence,

$$\theta(A)(x) = \bigcup_{a \in X} \{\theta(\delta_a)(x) \wedge \theta(\sigma_{A(a)}(x)\}.$$

By the definition of θ_x and by lemma 2.2 we infer

$$\theta(A)(x) = \bigcup_{a \in X} \{\delta_{s^{-1}(a)}(x) \wedge \theta_x(A(a))\} = \theta_x(A(s(x)))$$

for some permutation s^{-1} on X. The proof is over. ∎

The following theorem describes all automorphisms of a de Morgan algebra M(X).

Theorem 2.2. A mapping $\theta: \underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ is an automorphism of M(X) iff there is a family $\{\theta_x\}$, $x \in X$ of automorphisms of $[0,1]$ fulfilling the equation

$$\theta_x(\alpha) + \theta_x(1-\alpha) = 1 \qquad (2.2)$$

for all $x\in X, \alpha\in[0,1]$, and a permutation $s: X \to X$ such that

$$\theta(A)(x) = \theta_x(A(s(x))) \qquad (2.3)$$

for all $A\in \underset{\sim}{P}(X)$.

Proof. A mapping θ defined by (2.3) is an automorphism of $L(X)$. To prove that it is an automorphism of $M(X)$ it suffices to show that $\theta(\bar{A}) = \overline{\theta(A)}$. We have

$$\theta(\bar{A})(x) = \theta_x(1 - A(s(x))) = 1 - \theta_x(A(s(x))) =$$

$$1 - \theta(A)(x) = \overline{\theta(A)}(x)$$

by (2.2) and (2.3).

Conversely, let θ be an automorphism of $M(X)$. Then, by theorem 2.1, θ is represented by (2.3). Let us prove (2.2) in this case. We have $\theta(\bar{A}) = \overline{\theta(A)}$ for any $A\in \underset{\sim}{P}(X)$. Let $A = \sigma_\alpha$. Then $\bar{A} = \sigma_{1-\alpha}$ and we obtain $\theta(\sigma_{1-\alpha}) = 1 - \theta(\sigma_\alpha)$, or, by (2.3), $\theta_x(1-\alpha) = 1-\theta_x(\alpha)$, Q.E.D. ■

3. Automorphism groups.

Only lattices $L(X)$ are considered in this section, because all statements concerning automorphisms of these lattices are easily extended to the case of de Morgan algebras by theorem 2.2.

The set of all automorphisms of a given algebra is a group with respect to a composition of automorphisms. We use a symbol o as a denotation for any composition operation. Aut(L), Aut(X) and Aut([0,1]) denote, respectively, automorphism groups of a lattice $L(X)$, a set X and an interval $[0,1]$. X is supposed to be a finite set with cardinality n. Aut(X) is a symmetric group S_n and Aut([0,1]) is an automorphism group of the unit interval considered as a lattice with universal bounds (the latter group is studied in [6]).

By theorem 2.1 any automorphism of L(X) is determined by a pair $<\{\theta_x\}, s>$ where $\theta_x \epsilon Aut([0,1])$ for all $x \epsilon X$ and $s \epsilon Aut(X)$. The composition law in Aut(L) is given by

$$<\{\theta'_x\}, s'> \circ <\{\theta''_x\}, s''> = <\{\theta'_x \circ \theta''_{s'(x)}\}, s' \circ s''> .$$

For instance,

$$<\{\theta_x\}, s>^{-1} = <\{\theta^{-1}_{s^{-1}(x)}\}, s^{-1}>$$

and an identity element id_L in Aut(L) is $<\{id_{[0,1]}\}, id_X>$ where $id_{[0,1]}$ and id_X are identity elements in Aut([0,1]) and Aut(X), respectively.

Let us denote $K = \{<\{\theta_x\}, id_X>\}$ and $H = \{<\{id_{[0,1]}\}, s>\}$. It is easy to verify that K and H are subgroups of Aut(L) such that $K \cong Aut^n([0,1])$ and $H \cong Aut(X)$. Moreover, we have the following

Theorem 3.1. The group Aut(L) is a semidirect product of K by H.

Proof. Obviously, $K \cap H = \{id_L\}$. Hence, it suffices to prove that K is a normal subgroup and $K \cup H = Aut(L)$ (see theorem 6.5.3 in [4]).

We have

$$<\{\theta_x\}, s>^{-1} \circ <\{\theta'_x\}, id_X> \circ <\{\theta_x\}, s> =$$

$$<\{\theta^{-1}_{s^{-1}(x)} \circ \theta'_{s^{-1}(x)} \circ \theta_{s^{-1}(x)}\}, id_X> .$$

Hence, K is a normal subgroup.

Further,

$$<\{\theta_x\},\ s> = <\{\theta_x\},\ id_X> \circ <\{id_{[0,1]}\},s>.$$

Hence, K and H generate Aut(L). The proof is over. ■

The following definition introduces some particular automorphisms which are important in applications to representations of synonymy and antonymy.

<u>Definition 3.1.</u> 1) An automorphism θ of L(X) is said to be an <u>S-automorphism</u> if $\theta(A) = A$ for any crisp set A;

2) An automorphism θ of L(X) is said to be an <u>A-automorphism</u> if θ^2 is an S-automorphism and there is a crisp set A such that $\theta(A) \neq A$.

The following theorem yields a description of S- and A-automorphism

<u>Theorem 3.2.</u> An automorphism $\theta = <\{\theta_x\},\ s>$ of L(X) is an S-automorphism (resp. A-automorphism) iff $s = id_X$ (resp. $s^2 = id_X$ and $s \neq id_X$).

<u>Proof.</u> 1) Let $s = id_X$. Then $\theta(A) = A$ for any crisp set A, by theorem 2.1. Conversely, let $\theta(A) = A$ for any crisp set A. We have

$$\delta_a(x) = \theta(\delta_a)(x) = \delta_a(s(x)) = \delta_{s^{-1}(a)}(x).$$

Hence, $s(a) = a$ for all $a \in X$.

2) Let $s^2 = id_X$ and $s \neq id_X$. Then $\theta^2 = <\{\theta_x \circ \theta_{s(x)}\},\ id_X>$

is an S-automorphism by previous arguments. Obviously, there is a crisp set A such that $\theta(A) \neq A$ for $s \neq id_X$. Conversely, let θ^2 is an S-automorphism and there is a crisp set A such that $\theta(A) \neq A$.

We have $\theta^2 = \langle\{\theta_x \circ \theta_{s(x)}\}, s^2\rangle$ which implies $s^2 = id_X$, by previous arguments. Finally, $s \neq id_X$, since $\theta(A) \neq A$.

Corollary. The set of all S-automorphisms is a subgroup K.

Permutations s such that $s^2 = id_X$ will be called symmetries.

We define below a special class of subgroups of Aut(L), namely, SA-subgroups. If G is an SA-subgroup, then elements of G may be regarded as representations of synonymy and antonymy. The following definition is based on an observation that a synonym of a synonym is a synonym again and an antonym of an antonym should be a synonym.

Definition 3.2. A subgroup $G \subseteq \mathrm{Aut}(L)$ is said to be an SA-subgroup if

1) any element of G is either an S-automorphism or an A-automorphism;
2) G contains at least one A-automorphism;
3) composition of any two A-automorphisms is an S-automorphism.

The structure of SA-subgroups is established in the following

Theorem 3.3. Let G be an SA-subgroup of Aut(L). Then

1) there is a symmetry s such that

$$G \cap H = \{\langle \{id_{[0,1]}\}, id_X\rangle, \langle \{id_{[0,1]}\}, s\rangle\} \cong Z_2;$$

2) G is a semidirect product of $G \cap K$ by $G \cap H$.

Proof. 1) Let $\langle\{id_{[0,1]}\}, s_1\rangle$ and $\langle\{id_{[0,1]}\}, s_2\rangle$ are in $G \cap H$ and different from id_L. Then they are A-automorphisms, i.e. $s_1^2 = s_2^2 = id_X$. On the other hand $s_1 o s_2 = id_X$, by definition 3.2.3). Hence, $s_1 = s_1^{-1} = s_2$, i.e. $G \cap H$ contains only one automorphism, say $\langle\{id_{[0,1]}\}, s\rangle$, which is different from the identity element.

We have $G \cap H \cong Z_2$, for $s^2 = id_X$.

2) $G \cap K$ is a subgroup of G. Moreover, G is generated by $G \cap K$ and $G \cap H$. Indeed, any element $<\{\theta_x\}, s>$ of G is an S- or an A-automorphism, i.e. $s^2 = id_X$. We have

$$<\{\theta_x\}, s> = <\{\theta_x\}, id_X> \circ <\{id_{[0,1]}\}, s>$$

where $<\{\theta_x\}, id_X> \in G$, since $<\{id_{[0,1]}\}, s> \in G$. Hence, $G=(G\cap K)\cup(G\cap H)$. The rest of the proof is the same as the proof of theorem 3.1.

Corollary. G is a union of a normal subgroup G K of S-automorphisms and a unique coset of all A-automorphisms in G.

There are two special kinds of SA-subgroups which are useful in applications.

Definition 3.3. An SA-subgroup is said to be

1) a full SA-subgroup if $K \subseteq G$;
2) a homogeneous SA-subgroup if $G \cap K$ is a diagonal in $K \cong Aut^n([0,1])$.

Note, that G is a homogeneous SA-subgroup if and only if

$G \cap K = \{<\{\theta_x\}, id_X> \mid \theta_x \equiv \theta$ for some $\theta \in Aut([0,1])\}$.

Theorem 3.4. Any homogeneous SA-subgroup G of Aut(L) is isomorphic to a direct product of Aut([0,1]) on Z_2.

Proof. G is a semidirect product of $G \cap K \cong Aut([0,1])$ by $G \cap H \cong Z_2$, by the previous theorem. Let $<\{\theta\}, id_X> \in G \cap K$ and $<\{id_{[0,1]}\}, s> \in G \cap H$. We have

$$<\{\theta\}, id_X> \circ <\{id_{[0,1]}\}, s> = <\{id_{[0,1]}\}, s> \circ <\{\theta\}, id_X>,$$

i.e. any two elements of $G \cap K$ and $G \cap H$ commute. Hence, G is a direct product of $G \cap K$ on $G \cap H$ (see section 6.5 in [4]).

Corollary. Any element $<\{\theta\}, s>$ in a homogeneous SA-subgroup has a unique representation as a composition

$$<\{\theta\}, s> = <\{\theta\}, id_X> \circ <\{id_{[0,1]}\}, s> = <\{id_{[0,1]}\}, s> \circ <\{\theta\}, id_X> .$$

4. Representations by ultrafuzzy sets.

Let A be a given fuzzy set and θ an S-automorphism. We consider $B = \theta(A)$ as a synonym of A and define a degree of synonymity of B with respect to A by

$$\Sigma_A(B) = 1 - d(A,B) \qquad (4.1)$$

where d is any normed distance function on $\underset{\sim}{P}(X)$. We set $\Sigma_A(B) = 0$ iff B is not a synonym of A. $\Sigma_A(B)$ thus defined may be regarded as a value of a membership function of an ultrafuzzy set Σ_A (an ultrafuzzy set is a fuzzy set with universe $\underset{\sim}{P}(X)$). This set is considered as a fuzzy set of all synonyms of a given fuzzy set A.

We have $\Sigma_A(A) = 1$ which implies $\bigcup_{A\in\underset{\sim}{P}(X)} \Sigma_A = \underset{\sim}{P}(X)$. Hence, the family $\{\Sigma_A\}_{A\in\underset{\sim}{P}(X)}$ is a covering of $\underset{\sim}{P}(X)$. Actually, this covering is a partition of $\underset{\sim}{P}(X)$ if a max-Δ composition law is employed in ultrafuzzy set theory, where Δ is a connective defined by

$$x \,\Delta\, y = \max(x+y-1,\ 0).$$

(See [5] for definitions of coverings, partitions and related results and [2] and [3] where a detailed study of a connective Δ may be found). Then a resemblance relation

$$I(A,B) = \bigvee_{C\in\underset{\sim}{P}(X)} \{\Sigma_C(A) \,\Delta\, \Sigma_C(B)\}$$

generated by the covering $\{\Sigma_A\}_{A \in \underset{\sim}{P}(X)}$ is a similarity relation. Simple calculations yield

$$I(A,B) = \begin{cases} 1 - d(A,B), & \text{if } A \text{ and } B \text{ are synonyms,} \\ 0, & \text{otherwise} \end{cases}$$

The relation I may be regarded as a synonymity relation on $\underset{\sim}{P}(X)$. Classes of this similarity relation are ultrafuzzy sets of synonyms.

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