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TRIANGLE FUNCTIONS AND COMPOSITION OF PROBABILITY DISTRIBUTION FUNCTIONS (*)

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ABSTRACT

The equations of left and right distributivity of composition of distribution functions over triangle functions are solved in a restricted domain.

Let D^+ be the set of all probability distribution functions of non-negative random variables, i.e.,

$D^+=\{F\,|\,F:[-\infty,\infty]\to[0,1],\ F(0)=0,$ F is non-decreasing and left-continuous on $[-\infty,\infty)\}$,

and let D_1^+ and $D_{1,i}^+$ be the subspaces of D^+ defined by

$D_1^+=\{F\,|\,F\in D^+,\ F(1)=1\}$,

$D_{1,i}^+=\{F\,|\,F\in D_1^+,$ F is strictly increasing on $F^{-1}(0,1)\}$.

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Note that both D_1^+ and $D_{1,i}^+$ are closed under composition, i.e., if F and G are in D_1^+, then the function $F \circ G$ defined by $(F\circ G)(x)=F(G(x))$ is in D_1^+, and similarly for $D_{1,i}^+$.

Among the elements of D^+ are functions ε_a and A_b^a respectively defined, for $a \geqslant 0$ and $b \in [0,1]$, by:

$$\varepsilon_a(t) = \begin{cases} 0, & t \leqslant a, \\ 1, & t > a, \end{cases} \qquad A_b^a(t) = \begin{cases} 0, & t \leqslant 0, \\ b, & 0 < t \leqslant a, \\ 1, & a < t. \end{cases}$$

A mapping τ from $D^+ \times D^+$ into D^+ is a <u>triangle function</u> if the following conditions are satisfied for all F,G,H and K in D^+:

(i) $\tau(F,\varepsilon_0) = F$,
(ii) $\tau(F,G) \leqslant \tau(H,K)$ whenever $F \leqslant H$, $G \leqslant K$,
(iii) $\tau(F,G) = \tau(G,F)$,
(iv) $\tau(\tau(F,G),H) = \tau(F,\tau(G,H))$.

Triangle functions have been studied in detail in connection with triangle inequalities for probabilistic metric spaces (see [2,3,4]).

Our chief concern in this paper is to solve the following functional equations:

(I) $\tau(F\circ H,\ G\circ H) = \tau(F,G)\circ H$, for all $F,G,H \in D_1^+$,

(II) $\tau(H\circ F,\ H\circ G) = H\circ\tau(F,G)$, for all $F,G,H \in D_1^+$,

where, in each case τ is a triangle function to be found. Such equations arise in the investigation of isomorphisms of probabilistic metric spaces and are also of interest for the techniques used in their solutions.

In this paper we let Z denote the set of continuous t-norms, i.e., the set of continuous binary operations T on [0,1] which are commutative, associative, non-decreasing in each place with 1 as a unit and 0 as a null element. Correspondingly, L_0 will denote the set of two-place functions L from $[0,\infty]\times[0,\infty]$ onto $[0,\infty]$ which are commutative, associative, continuous on the domain, increasing in each place, and have 0 as a unit.

A triangle function τ will be called L-strict if it is continuous with respect to the modified Lévy metric (see [3]) and there exists L in L_o such that $\tau(\varepsilon_x,\varepsilon_y) = \varepsilon_{L(x,y)}$, for all x and y in $[0,+\infty]$.

Any element T in Z induces the triangle function π_T defined by

$$\pi_T(F,G)(x) = T(F(x),\ G(x)).$$

For T in Z and L in L_o, we define the L-strict triangle function $\tau_{T,L}$ by

$$\tau_{T,L}(F,G)(x) = \begin{cases} 0 & , \quad x \leqslant 0, \\ \sup\{T(F(u),G(v))\,|\,L(u,v)=x\}, & x > 0. \end{cases}$$

We note first that π_T is a solution of (I) for every T in Z. Our immediate aim is to show that π_T is the general solution of (I), i.e., that we have:

Theorem 1. A continuous triangle function τ is a solution of (I) if and only if there exists a t-norm T such that $\tau=\pi_T$.

To prove Theorem 1, we begin with two lemmas.

Lemma 1. If a continuous triangle function τ satisfies (I), then

$$\tau(\varepsilon_x,\varepsilon_y) = \varepsilon_{Max(x,y)}, \text{ for all } x,y \in [0,1].$$

Proof. For $x,y \in (0,1)$, define $H_{x,y} \in D_1^+$ by:

$$H_{x,y}(t) = \begin{cases} 0, & t \leqslant 0, \\ \frac{|x-y|}{Max(x,y)}\, t + Min(x,y), & 0 < t \leqslant Max(x,y), \\ t, & Max(x,y) \leqslant t \leqslant 1, \\ 1, & 1 \leqslant t. \end{cases}$$

Then,

$$\varepsilon_{Min(x,y)} \circ H_{x,y} = \varepsilon_o \text{ and } \varepsilon_{Max(x,y)} \circ H_{x,y} = \varepsilon_{Max(x,y)},$$

and using (I) we immediatly obtain

$$\varepsilon_{Max(x,y)} = \tau(\varepsilon_{Max(x,y)}, \varepsilon_o) = \tau(\varepsilon_{Max(x,y)} \circ H_{x,y}, \varepsilon_{Min(x,y)} \circ H_{x,y})$$

$$= \tau(\varepsilon_x, \varepsilon_y) \circ H_{x,y}.$$

Thus $\tau(\varepsilon_x, \varepsilon_y)(Max(x,y))=0$, and whenever $t>Max(x,y)$, we have

$$\tau(\varepsilon_x, \varepsilon_y)(t) = (\tau(\varepsilon_x, \varepsilon_y) \circ H_{x,y})(t) = \varepsilon_{Max(x,y)}(t) = 1.$$

If $x=y=1$, we have

$$\tau(\varepsilon_1, \varepsilon_1) = \tau(\lim_{n\to\infty} \varepsilon_{1-1/n}, \lim_{n\to\infty} \varepsilon_{1-1/n}) = \lim_{n\to\infty} \tau(\varepsilon_{1-1/n}, \varepsilon_{1-1/n})$$

$$= \lim_{n\to\infty} \varepsilon_{1-1/n} = \varepsilon_1.$$

Analogously when $x=1$ and $y<1$ it follows $\tau(\varepsilon_y, \varepsilon_1) = \varepsilon_1$.

Lemma 2. If a continuous triangle function τ satisfies (I), then there exists $T \in Z$ such that for all $x,y \in (0,1)$, we have

$$\tau(A_x^{1/2}, A_y^{1/2}) = A_{T(x,y)}^{1/2} \quad (1)$$

Proof. First note that for any $z \in (0,1)$ we have $\varepsilon_{1/2} \leqslant A_z^{1/2} \leqslant \varepsilon_o$, so that it follows from the preceding lemma that

$$\varepsilon_{1/2} = \tau(\varepsilon_{1/2}, \varepsilon_{1/2}) \leqslant \tau(A_x^{1/2}, A_y^{1/2}) \leqslant \tau(\varepsilon_o, \varepsilon_o) = \varepsilon_o.$$

Now let a,b be any numbers in $(0,1/2)$ such that $a<b$. Let $K_{a,b} \in D_1^+$ be given by:

$$K_{a,b}(t) = \begin{cases} 0, & t \leqslant 0, \\ \frac{b}{a}\, t, & 0 < t \leqslant a, \\ b, & a \leqslant t \leqslant 1/2, \\ 1, & 1/2 < t. \end{cases}$$

Then we have $A_x^{1/2} \circ K_{a,b} = A_x^{1/2}$ for all $x\in(0,1)$, whence

$$\tau(A_x^{1/2}, A_y^{1/2})(a) = \tau(A_x^{1/2} \circ K_{a,b}, A_y^{1/2} \circ K_{a,b})(a)$$

$$= \tau(A_x^{1/2}, A_y^{1/2})(K_{a,b}(a)) = \tau(A_x^{1/2}, A_y^{1/2})(b),$$

i.e., $\tau(A_x^{1/2}, A_y^{1/2})$ is constant on $(0,1/2)$. Thus if we define

$$T(x,y) = \tau(A_x^{1/2}, A_y^{1/2})(1/2), \qquad (2)$$

(1) follows and a short computation shows that $T\in\mathcal{Z}$.

Now we can solve (I) completely.

Proof of Theorem 1. Assume that τ satisfies (I) and consider T as defined by (2). We need to show that for all $F,G\in D_1^+$ and for any $x>0$ we have

$$\tau(F,G)(x) = T(F(x), G(x)). \qquad (3)$$

Obviously, if $x>1$, then $F(x)=G(x)=1$, and (3) follows from Lemma 1. So let $x\in(0,1]$. In this case,

$$F \circ A_x^{1/2} = A_{F(x)}^{1/2} \text{ and } G \circ A_x^{1/2} = A_{G(x)}^{1/2},$$

whence, by (I) and Lemma 2, there is a T in $\mathcal{Z}$ such that:

$$\tau(F,G)(x) = \tau(F,G)(A_x^{1/2}(1/2)) = (\tau(F,G) \circ A_x^{1/2})(1/2)$$

$$= \tau(F \circ A_x^{1/2}, G \circ A_x^{1/2})(1/2)$$

$$= \tau(A_{F(x)}^{1/2}, A_{G(x)}^{1/2})(1/2) = A_{T(F(x),G(x))}^{1/2}(1/2)$$

$$= T(F(x), G(x)),$$

and the theorem is proved.

We turn now our attention to (II). We recall that the duality theorem of [1] shows that the L-strict triangle function $\tau_{M,L}$ (where $M(x,y) = \text{Minimum}(x,y)$, and $L \in L_o$) admits the representation

$$[\tau_{M,L}(F,G)]^{\wedge} = L(F^{\wedge},G^{\wedge}), \qquad (4)$$

where for any F in D^+, $F^{\wedge}$ denotes the quasi-inverse of F (see [1]).

<u>Lemma 3.</u> $\tau_{M,L}$ is a solution of (II).

Proof. If F,G and H are in D_1^+ then we have, for all $x>0$,

$$\begin{aligned}\tau_{M,L}(H\circ F,\ H\circ G)(x) &= \sup_{L(u,v)=x} M(H(F(u)),\ H(G(v)))\\ &= \sup_{L(u,v)=x} H(M(F(u),G(v)))\\ &= H\Big(\sup_{L(u,v)=x} M(F(u),G(v))\Big)\\ &= (H\circ\tau_{M,L}(F,G))(x),\end{aligned}$$

where in the first and third equalities we have used, respectively, the fact that H is non-decreasing and left-continuous.

We note that if an L-strict triangle function τ satisfies (II), then for all $x,y \in (0,1)$ we have,

$$\tau(\varepsilon_x \circ F,\ \varepsilon_x \circ G) = \varepsilon_x \circ \tau(F,G), \qquad (5)$$

and

$$\tau(H \circ \varepsilon_x,\ H \circ \varepsilon_y) = H \circ \tau(\varepsilon_x,\varepsilon_y), \qquad (6)$$

for all F,G and H in D_1^+.

Now we can solve (II).

<u>Theorem 2.</u> A triangle function τ which is L-strict is a solution of (II) if and only if $\tau=\tau_{M,L}$ on $D_1^+ \times D_1^+$.

Proof. First we note that if τ is L-strict then L can be defined by

$$L(a,b) = \tau(\varepsilon_a,\varepsilon_b)^{\wedge}(1/2), \qquad (7)$$

for all $a,b\geq 0$. We need to show that whenever τ satisfies (II) then $\tau=\tau_{M,L}$. In order to do this, we remark that when $H\in D^+_{1,i}$ then the right quasi-inverse $H^\vee$ is equal to the left quasi-inverse $H^\wedge$ because $H^\wedge$ is continuous. In this case, $\varepsilon_x\circ H = \varepsilon_{H^\vee(x)} = \varepsilon_{H^\wedge(x)}$ for all $x\in(0,1)$. Then for any F,G in $D^+_{1,i}$ and $x\in(0,1)$, (5) and (7) yield

$$\begin{aligned}\tau(F,G)^\vee(x) &= [\varepsilon_{\tau(F,G)^\vee(x)}]^{\wedge}(1/2) = [\varepsilon_x\circ\tau(F,G)]^{\wedge}(1/2)\\ &= \tau(\varepsilon_x\circ F,\varepsilon_x\circ G)^{\wedge}(1/2)\\ &= \tau(\varepsilon_{F^\wedge(x)},\varepsilon_{G^\wedge(x)})^{\wedge}(1/2) = L(F^\wedge(x),G^\wedge(x))\\ &= [\tau_{M,L}(F,G)]^{\wedge}(x) = [\tau_{M,L}(F,G)]^\vee(x),\end{aligned}$$

i.e., $\tau=\tau_{M,L}$ on $D^+_{1,i}\times D^+_{1,i}$. Using the continuity of τ it is possible to extend this conclusion to the space $D^+_1\times D^+_1$ because any function F in D^+_1 can be obtained as the weak limit of a sequence of distribution functions in $D^+_{1,i}$. The theorem is proved.

To end this paper we can show that the conditions assumed in the above theorem are natural.

<u>Theorem 3.</u> If a triangle function τ satisfies (II) then there exists a function L from (0,1)x(0,1) into $[0,+\infty]$ such that $\tau(\varepsilon_a,\varepsilon_b)=\varepsilon_{L(a,b)}$, for all $a,b\in(0,1)$.

Proof. If $H\in D^+_1$ then $H\circ\varepsilon_x=\varepsilon_x$, for all $x\geq 0$. Thus if $a,b\in(0,1)$ and τ satisfies (II) we have by (6)

$$\tau(\varepsilon_a,\varepsilon_b)=\tau(H\circ\varepsilon_a,\ H\circ\varepsilon_b)=H\circ\tau(\varepsilon_a,\varepsilon_b), \qquad (8)$$

for all $H\in D^+_1$. If there were a point $x_o>0$ such that $\tau(\varepsilon_a,\varepsilon_b)(x_o)\in(0,1)$, the upon replacing H in (8) by the uniform distribution function

$$U_{o,\tau(\varepsilon_a,\varepsilon_b)(x_o)}(t)=\begin{cases}0, & t\leq 0\\ t/\tau(\varepsilon_a,\varepsilon_b)(x_o), & 0\leq t\leq\tau(\varepsilon_a,\varepsilon_b)(x_o),\\ 1, & t\geq\tau(\varepsilon_a,\varepsilon_b)(x_o),\end{cases}$$

we would obtain

$$\tau(\varepsilon_a,\varepsilon_b)(x_o)=(U_{o,\tau(\varepsilon_a,\varepsilon_b)(x_o)}o\tau(\varepsilon_a,\varepsilon_b))(x_o) = 1,$$

which is a contradiction. Thus the range of $\tau(\varepsilon_a,\varepsilon_b)$ must be $\{0,1\}$, i.e., there is a constant $k\geq 0$ such that $\tau(\varepsilon_a,\varepsilon_b)=\varepsilon_k$. Using the fact that $\varepsilon_k^{\wedge}(1/2)=k$, we can define L from $(0,1)\times(0,1)$ into $[0,\infty]$ by $L(a,b)=\tau(\varepsilon_a,\varepsilon_b)^{\wedge}(1/2)$. Then the above argument shows that the theorem holds.

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