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A METHOD FOR THE CALCULUS OF BERNSTEIN'S POLYNOMIAL

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ABSTRACT

A systematic method for the calculus of Bernstein's polynomial is described. It consists on reducing the problem to a homogeneous linear system of equations that may be constructed by fixed rules. Several problems about its computer implementation are discussed.

0. Introduction.

For any $f \in C[x_1,\dots,x_n]$ there exist a polynomial B(s) and a differential operator P(x,s,D) that satisfy

$$P(x,s,D)\ f^s = B(s)\ f^{s-1} \qquad (1)$$

The set of B's which are solution of the preceding equation is an ideal. We note b(s) (Bernstein's polynomial of f) its generator. An excellent report about its interest and properties is available at Jean-Michel BONY, "Polynômes de Bernstein et monodromie (d'après B. Malgrange)", Lecture Notes in Mathematics, 514, Springer Verlag, 1976. A main result is that the roots of b are all rational and less than one (Malgrange).

Let $a,b,i\in N^n;\alpha,\omega:N\to N$; and $z\in K^n$. I will use the notation

a) $z^i = z_1^{i_1}\ldots z_n^{i_n}$

b) $|i| = \sum_{k=1}^{n} i_k$

c) $\sum_{i=\alpha(a)}^{\omega(b)}$ is the sum over all $\alpha(a_k)\leqslant i_k\leqslant\omega(b_k)$

that hold $\alpha(|a|)\leqslant|i|\leqslant\omega(|b|)$. In the case where α and ω are constants this is equivalent to the usual multiindex sum $\sum_{\alpha\leqslant|i|\leqslant\omega}$

Using the above conventions f and P can be written as

$$f(x) = \sum_{i_x=0}^{g_f} f_{i_x} x^{i_x}$$

$$P(x,s,D) = \sum_{i_x=0}^{g_x}\sum_{i_s=0}^{g_s}\sum_{\substack{i_d=0\\|i_d|\neq 0}}^{g_d} a_{i_x i_s i_d}\, x^{i_x} s^{i_s} D^{i_d}$$

1. Derivating f^s.

The expression obtained applying D^{i_d} to f^s has the form

$$D^{i_d}f^s = \sum_{k=1}^{|i_d|}\sum_{i_x=0}^{kg_f-|i_d|} c_{i_x k i_d}\, x^{i_x}\,\psi_k\, f^{s-k}$$

where $\psi_k = \prod_{i=0}^{k-1}(s-i) = \sum_{\ell=1}^{k}\pi_{k\ell}\, s^{\ell}$ (by definition of π's).

This can easily be proved by induction on i_d.

The step of induction also provides an algorithm for the calculus of c's (for $|i_d| > 1$) because the first equality and

$$D_1^{i_{d_1}} \ldots D_m^{i_{d_m}} \ldots D_n^{i_{d_n}} f^s = \sum_{k=1}^{|i_d|-1} \; \sum_{i_x=0}^{kg_f-(|i_d|-1)-1}$$

$$(i_{x_m}+1)\, c_{i_{x_1}\ldots i_{x_m}+1\ldots i_{x_n}\, k\, i_{d_1}\ldots i_{d_m}-1\ldots i_{d_n}}\; x^{i_x}\, \psi_k\, f^{s-k}$$

$$+ \sum_{k=1}^{|i_d|-1} \; \sum_{i_x=0}^{kg_f-(|i_d|-1)} \; \sum_{i'_x}^{g_f-1} c_{i_x k\, i_{d_1}\ldots i_{d_m}-1\ldots i_{d_n}}\; c_{i'_x 1\, 0\ldots 1 \overset{m}{\ldots} 0}$$

$$x^{i_x+i'_x}\, \psi_{k+1}\, f^{s-(k+1)}$$

relate the $c_{i_x k\, i_{d_1}\ldots i_{d_m}\ldots i_{d_n}}$'s to the $c_{i_x k\, i_{d_1}\ldots i_{d_m}-1\ldots i_{d_n}}$'s

and the $c_{i_x k\, 0\ldots 1 \overset{m}{\ldots} 0}$'s.

This assumes the previous evaluation of the first order derivatives:

$$D_m f^s = s f^{s-1} D_m f = \sum_{i_x=0}^{g_f-1} c_{i_x 1\, 0\ldots 1 \overset{m}{\ldots} 0}\; x^{i_x}\, \psi_1\, f^{s-1}$$

$$= \sum_{i_x=0}^{g_f-1} (i_{x_m}+1)\, f_{i_{x_1}\ldots i_{x_m}+1\ldots i_{x_n}}\; x^{i_x}\, \psi_1\, f^{s-1}$$

2. Developing Pf^s.

Applying P to f^s and rearranging its terms the following expression is obtained:

$$Pf^s = \sum_{k=1}^{g_d} \sum_{i=1}^{g_s+k} \Lambda_{ki}\, s^i\, f^{s-k}$$

where

$$\Lambda_{ki} = \sum_{i_x=0}^{g_x+k(g_f-1)} x^{i_x} \sum_{\substack{i_d=0 \\ |i_d| \geq k}}^{g_d} \sum_{\ell_s=\max(1,i-g_s)}^{\min(k,i)}$$

$$\sum_{\ell_x=\max(0,i_x-g_x)}^{\min(i_x,kg_f-i_d)} a_{i_x-\ell_x,\, i_s-\ell_s,\, i_d}\; c_{\ell_x k\, i_d}\; \pi_{k\,\ell_s} \qquad (2)$$

By its generation Λ_{ki} are only defined for $1 \leqslant i \leqslant g_s+k$. In the following Λ_{ki} also denotes the null polynomial if $i > k+g_s$ $(1 \leqslant k \leqslant g_d)$.

3. Reducing Pf^s to the form $B(s)f^{s-1}$.

The coefficients $a_{i_x i_s i_d}$ must be such that Pf^s may be reduced to the form $B(s)f^{s-1}$. The system that expresses all the possible ways of simplification is

$$\begin{cases} \Lambda_{g_d i} = \sum_{h=1}^{g_d-1} R_{g_d i h}\, f^h & (1 \leqslant i \leqslant g_s+g_d) \\ \Lambda_{ki} + \sum_{h=1}^{g_d-k} R_{k+h,i,h} = \sum_{h=1}^{k-1} R_{kih}\, f^h & (1 < k < g_d;\ 1 \leqslant i \leqslant g_s+g_d) \\ \Lambda_{1i} + \sum_{h=1}^{g_d-k} R_{k+h,i,h} = H_i & (1 \leqslant i \leqslant g_s+g_d) \end{cases}$$

where $R_{\ell ih}$ are polynomials in x and H_i are polynomials of degree zero whose independent term is b_i, coefficient of degree i of the searched Bernstein's polynomial (or any of its multiples).

Definition of the aggregates $R_{ki} = \sum_{\ell=k}^{g_d} \sum_{h=\ell-k+1}^{\ell-1} R_{\ell ih} f^{h-(\ell-k)-1}$

allows the reduction of the preceding system to

$$\begin{cases} \Lambda_{g_d i} = f R_{g_d i} & (1 \leq i \leq g_s+g_d) \\ \Lambda_{k\,i} + R_{k+1,i} = f R_{ki} & (1\leq k\leq g_d;\ 1\leq i\leq g_s + g_d) \\ \Lambda_{1\,i} + R_{2\,i} = H_i & (1\leq i\leq g_s+g_d) \end{cases} \qquad (3)$$

It is also possible to prove by induction on k (from g_d to 1) that the degree of R_{ki} is less or equal to $g_k+k(g_f-1)-g_f$. The discussion of the degree of R_{ki} allows to establish the form of fR_{ki}:

$$fR_{ki} = \sum_{k_x=0}^{g_x+k(g_f-1)} x^{k_x} \sum_{\ell_x=\max(0,k_x-(g_x+k(g_f-1)\ g_f))}^{\min(g_f,k_x)} f_{\ell_x} r_{k,i,k_x-\ell_x}.$$

A simple accounting leads then to establish that this system has

$$(g_s+g_d) \sum_{k=1}^{g_d} \binom{n+g_k+k(g_f-1)}{n} \quad \text{equations and}$$

$$g_s+g_d+(g_s+1)\binom{n+g_x}{n}\left[\binom{n+g_d}{n}-1\right] + (g_s+g_d) \sum_{k=1}^{g_d} \binom{n+g_x+k(g_f-1)-g_f}{n}$$

unknowns. (There are three types of unknowns: the coefficients b_i, those of P and those of R's).

The coefficients of (3) are zeros, ones, those of f and those obtained from (2). They can be placed systematically by an adequate numbering of unknowns.

4. Solving the System.

The system (3) is only derived from (1). But, as noticed earlier, the set of solutions of this equality is an ideal and we are only interested in its generator.

It would seem reasonable to expect that essaying values of g_s and g_d in a way that its sum is increased by no more than one at each step until (3) admits non-trivial solution ($B(s)\neq 0$) the first polynomial found is the one of smallest degree. But this is a thing that nobody has proved yet. The absence of results about that question leads to foresee the possibility that (3) can have as a result a family of multiples of b expressed in terms of a set of arbitrary constants. In this case we shall take a base of the subspace of solutions, calculate its roots and take as result the polynomial product of the binomials derived from its common roots. As we are dealing with multiples of Bernstein's polynomial, there is no guarantee that rhe found roots be rational. Those roots shall not be taken into consideration.

5. Concluding Remarks.

The method described is systematic enough to allow its computer implementation. In fact, there exists a Fortran program that is operative. As it uses exact rational arithmetic, exact rational roots of b are obtained. Unfortunately, the amount of memory and time necessary grows fast with g_f, g_x, g_s and n, being the bottleneck the solving of systens (3). This amount would be strikingly reduced if the caracteristics (g_s, g_x, g_d) of the operator to be essayed for a given f were known or at least bounded. The only bound known at present, that the degree of b is less than or equal to $\mu+1$ (μ is the Milnor's number of f), is merely a hint on g_s+g_d. For g_x there is no hint at all but experience shows that generally it is not too large.