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INVOLUTIONS IN FUZZY SET THEORY*

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To Lotfi Zadeh on the occasion of his 60^{th} birthday.

ABSTRACT

All possible involutions in fuzzy set theory are completely described. Any involution is a composition of a symmetry on a universe of fuzzy sets and an involution on a truth set.

The most widespread models for negations in fuzzy set theory are involutions (see [1] - [4]). Usually, some special kinds of involutions are considered as negations. All involutions in fuzzy set theory are fully described below.

Let X be a finite set and L - a complete distributive lattice with universal bounds 0 and I. Fuzzy sets with universe X are regarded here as mappings A: X → L. A lattice L is supposed to be a directly indecomposable, i.e. L is not represented as a direct product of non-trivial lattices. For example, any bounded chain is a directly indecomposable lattice and, therefore, classical fuzzy set theory is included in the framework considered.

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We denote $\underset{\sim}{P}(X) = L^X$ a set of all fuzzy sets with universe X. $\underset{\sim}{P}(X)$ is a complete distributive lattice with respect to an ordering induced by L. In particular, $\phi(x) \equiv 0$ and $X(x) \equiv I$ are universal bounds in $\underset{\sim}{P}(X)$. Let us denote $P(X) = 2^X$ a set of all crisp subsets of X. Then P(X) is a Boolean sublattice in the lattice $\underset{\sim}{P}(X)$.

The following lemma is based on [5], III, 8.

Lemma 1. P(X) is a center of $\underset{\sim}{P}(X)$.

Proof. Follows immediately from Theorem 11, [5], III, 8, because L is a directly indecomposable lattice.

Let $\theta: \underset{\sim}{P}(X) \to \underset{\sim}{P}(X)$ be an involution, i.e. a dual automorphism of period two ([5], p. 3). We denote $\bar{A} = \theta(A)$. It is easy to verify that

$$\overline{\cup A_i} = \cap \bar{A}_i, \text{ and} \qquad \overline{\cap A_i} = \cup \bar{A}_i \tag{1}$$

in $\underset{\sim}{P}(X)$.

Since the definition of a direct product is invariant under dual automorphisms, the restriction of θ on P(X) is an involution in P(X). The following construction gives a description of involutions in P(X). Let us denote

$$\delta_a(x) = \begin{cases} I, & x = a, \\ 0, & x \neq a. \end{cases} \tag{2}$$

Then δ_a are atoms in a Boolean lattice P(X). Let A' denote a complement of A in P(X). The $\theta(A') = \theta(A)'$ and $\phi: A \to \theta(A')$ is an automorphism of P(X). Hence, we have $\bar{A} = \phi(A)'$. It is evident that each automorphism of a finite Boolean lattice is determined by some permutation s of the set of atoms. In particular, we have

$$\bar{\delta}_a = \bigcup_{x \neq s(a)} \delta_x. \tag{3}$$

Since θ and a complementation in P(X) have period two, s is a symmetry on X, i.e. a permutation of period two.

We will describe now an important class of involutions in $\underset{\sim}{P}(X)$. Let s be a symmetry on X and $\Theta = \{\theta_x\}_{x\in X}$ - a family of dual automorphisms in L such that $\theta_{s(x)} = \theta_x^{-1}$. In particulary, θ_x is an involution if $s(x) = x$. Then the following formula defines an involution in $\underset{\sim}{P}(X)$

$$(4) \qquad \bar{A}(x) = \theta_x(A(s(x))).$$

Involutions, defined by (4), will be called decomposable involutions.

Theorem 1. Any involution in $\underset{\sim}{P}(X)$ is a decomposable involution, i.e. is defined by (4) for some s and Θ

Proof. Let us define a fuzzy set σ_α by

$$\sigma_\alpha(x) \equiv \alpha$$

for each $\alpha \in L$. Then any fuzzy set A can be represented as

$$A = \bigcup_a (\delta_a \cap \sigma_{A(a)}).$$

We have, by (1),

$$\bar{A} = \bigcap_a (\bar{\delta}_a \cup \bar{\sigma}_{A(a)}).$$

Let us denote $\theta_x(\alpha) = \bar{\sigma}_\alpha(x)$. Then, by (3),

$$(5) \qquad \bar{A}(x) = \bigwedge_a (\bar{\delta}_a(x) \vee \bar{\sigma}_{A(a)}(x)) = \theta_x(A(s(x)))$$

for some symmetry s. We have

$$\alpha \leqslant \beta \Leftrightarrow \sigma_\alpha \leqslant \sigma_\beta \Leftrightarrow \sigma_\alpha = \sigma_\alpha \cap \sigma_\beta \Leftrightarrow \bar{\sigma}_\alpha = \bar{\sigma}_\alpha \cup \bar{\sigma}_\beta \Leftrightarrow \bar{\sigma}_\alpha \geqslant \bar{\sigma}_\beta \Leftrightarrow \theta_x(\alpha) \geqslant \theta_x(\beta).$$

Hence, θ_x is a dual automorphism. For $A = \bar{\sigma}_\alpha$ formula (5) gives

$$\alpha = \theta_x(\theta_{s(x)}(\alpha))$$

i.e. $\theta_{s(x)} = \theta_x^{-1}$, Q.E.D.

If we would like to use involutions as a model for negations in fuzzy set theory it is natural to involve so-called Extension Principle, i.e. an assumption that a restriction of a negation on P(X) is a usual complementation. Then $s(x) = x$ and we obtain the following

Corollary 1. ([4]) Any involutionary negation $A \to \bar{A}$ in fuzzy set theory is defined by

$$\bar{A}(x) = \theta_x(A(x))$$

for some family $H = \{\theta_x\}_{x \in X}$ of involutions in L.

Note in conclusion that the assumption that L is a directly indecomposable lattice plays an essential role in our considerations. For example, let us consider the case when $L = 2^Y$, where Y is a two-point set, and let X be a two-point universe set. Then $L^X = \underset{\sim}{P}(X)$ is isomorphic to 2^Z, where Z is a four-element set. It is easy to calculate that there are 10 involutions in 2^Z, but only 8 decomposable involutions in $\underset{\sim}{P}(X)$. Hence, there are two involutions in $\underset{\sim}{P}(X)$ which are different from that defined by (4).

References.

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