Stochastica, Vol. IV nº 3 (1980)

A GENERAL DEDUCTION THEOREM

Salvatore Guccione(*) - Roberto Tortora(**)

ABSTRACT

In this paper we present a very general deduction theorem which - based upon a uniform notion of proof from hypotheses - holds for a very large class of logical systems. Most of the known results for classical and modal logics, as well as new results, are immediate corollaries of this theorem.

1. The usual formulation of the Deduction Theorem for a formal system, is the following:

(*) "If there is a proof from the hypotheses $A_1, A_2, \ldots, A_n$ for the formula B, then there is a proof from the hypotheses $A_1, A_2, \ldots, A_{n-1}$ for the formula $A_n \supset B$."

of course in every statement of the type (*) the informal notion of proof from hypotheses is involved

As we know, in classical logic, the notion of proof from hypotheses is independent from the particular system in which we are working. On the contrary there are logical systems - for instance modal systems - for which it seems that every attempt to prove a Deduction Theorem in the form (*) requires a formal definition ad hoc of proof from hypotheses. Zeman, for example, introduces different clauses depending on the specific system (see [1], p. 194/197).

In this paper we will follow a different approach: first we define a uniform notion of proof from hypotheses, second we demonstrate a General Weakened Deduction Theorem. This theorem holds for a very large class of systems. Finally we are able to derive, as particular cases of the theorem, the well-known Deduction Theorems for classical logic and some weakened Deduction Theorems for most important propositional and first-order modal systems.

A point of view similar for many respects to ours, is followed by Perzanowski [2],[3]. However some essential differences occur between our and Perzanowski's approach. For example he is interested only in modal propositional logic and he proves different theorems for several calculi. On the contrary we prove a unique general theorem for classical and modal (propositional and first-order) logics.

On the other hand Perzanowski is interested in sufficient and necessary conditions for a deduction theorem, while we investigate only sufficient conditions. Moreover, our use of the notion of 'to depend upon' allows some of our results to be a bit more refined than Perzanowski's.

2. Let S be an axiomatic formal system and R_S the set of the rules of inference of S.

We follow Mendelson's definitions of rule of inference and of direct consequence ([4], p.29).

Definition 1. A finite sequence $\beta_1,\dots,\beta_m$ of wfs is called a deduction or a proof from hypotheses Γ (where Γ is a set of wfs) in S of the wf β if and only if β is β_m and for each $i \leqslant m$: either β_i is an axiom of S, or β_i belongs to Γ, or β_i is a direct consequence by virtue of some rule of inference, of some of the preceding wfs in the sequence.

Definition 2. Let Γ be a set of wfs of S. Let $\beta_1, \beta_2,\dots,\beta_m$ be a deduction from Γ in S, together with a justificacion for each step of the deduction.

If α is a wf in Γ we say that β_i, $i \leqslant m$, depends upon α if and only if: either β_i is α and it occurs in the deduction as an element of Γ; or β_i is justified as a direct consequence, by virtue of some $\varphi \in R_S$, of some preceding wfs of the sequence and at least one of these formulas depends upon α.

For the sequel, we assume that the system S contains a symbol "$\supset$" for implication and satisfies:

I) $\vdash_S \alpha \supset (\beta \supset \alpha)$ for every α,β wfs of S;

II) $\vdash_S [\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)$ for every α,β,γ wfs of S;

III) Modus Ponens (MP) is a rule of inference of S.

No additional condition is required for S.

We list here some easy properties of every system S satisfying I/III. Γ is a set of wfs of S,α,β, γ_j, δ are wfs of S, r is a natural number $\geqslant 0$.

P1) If $\Gamma \vdash_S \alpha$ then $\Gamma \vdash_S \beta \supset \alpha$ for every β.

P2) $\vdash_S \alpha \supset \alpha$ for every α.

P3) Let $(i_1,\ldots,i_r)$ be a permutation of $(1,\ldots r)$.
If $\Gamma \vdash_S \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_r \supset \delta)\ldots)$, then $\Gamma \vdash_S \gamma_{i_1} \supset (\gamma_{i_2} \supset \ldots \supset (\gamma_{i_r} \supset \delta)\ldots)$.

P4) If $\Gamma \vdash_S \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_r \supset (\alpha \supset \beta)\ldots)$ and $\Gamma \vdash_S \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_r \supset \alpha)\ldots)$, then $\Gamma \vdash_S \gamma_1 \supset (\gamma_2 \supset \ldots \supset (\gamma_r \supset \beta)\ldots)$.

P5) If $\Gamma \vdash_S \alpha \supset \beta$ and $\Gamma \vdash_S \beta \supset \delta$, then $\Gamma \vdash_S \alpha \supset \delta$.

Definition 3. A rule of inference φ is said to be reinforcing (monotonic) if and only if φ has only one premise and it has the form $\dfrac{\alpha}{C_\varphi \alpha}$ ($\dfrac{\alpha \supset \beta}{C_\varphi \alpha \supset C_\varphi \beta}$; respectively), where α and β are wfs os S and the prefix C_φ - uniquely determined by φ - is a string of symbols from the language of S such that for each wf γ of S, $C_\varphi \gamma$ is a wf of S.

3. Let S be a system satisfying I/III and also the following two hypotheses:

IV) The only non-reinforcing and non-monotonic rule of inference of S is MP.

V) For every $\varphi \epsilon R_S (\varphi \neq$ MP), and for every wfs α,β of S the following holds:

$$\vdash_S C_\varphi(\alpha \supset \beta) \supset (C_\varphi\alpha \supset C_\varphi\beta).$$

Then it is possible to prove the following:

Theorem 1. Let Γ be a set of wfs of S,α,β wfs of S and suppose $\Gamma,\alpha \vdash_S \beta$.

Let $\varphi_1,\dots,\varphi_i,\dots,\varphi_k$ be the ordered list (possibly with repetitions) of the rules distinct from MP which are applied in the deduction to wfs depending upon α, and denote by $C_1,\dots,C_i,\dots,C_k$ their prefixes. Then there exists an integer $p \geqslant 0$ such that

$$(1) \qquad \Gamma \vdash_S C^{(1)}\alpha \supset (C^{(2)}\alpha \supset \dots \supset (C^{(p)}\alpha \supset \beta)\dots)$$

where every "multiple prefix" $C^{(t)}$ is obtained from the "total prefix" $C_k C_{k-1}\dots C_i \dots C_1$ cancelling some $(0,1,\dots,k)$ of the prefixes C_i.

Proof. Let $\beta_1,\beta_2,\dots,\beta_n = \beta$ be a deduction of β from Γ,α in S.

We prove by induction that $\forall i \leqslant n$

$$(2) \qquad \Gamma \vdash_S C_i^{(1)}\alpha \supset (C_i^{(2)}\alpha \supset \dots \supset (C_i^{(p_i)}\alpha \supset \beta_i)\dots)$$

where $\varphi_1,\varphi_2,\dots,\varphi_{k_i}$ is the ordered list of the rules ($\neq$MP) applied to wfs depending upon α in the deduction $\beta_1,\beta_2,\dots,\beta_i$. Moreover, every $C_i^{(t)}$ in (2) is obtained from the string $C_{k_i} C_{k_i-1}\dots C_1$ cancelling some prefixes.

We must distinguish several cases depending upon the justification for β_i in the deduction.

(i) β_i is an axiom of S, or $\beta_i \epsilon \Gamma$ or β_i is α. We can easily prove, using P1 and P2, that:

$$\Gamma \vdash_S \alpha \supset \beta_i ,$$

that is, (2) with $p_i=1$ and $C_i^{(p_i)}$ the null prefix.

(ii) β_i is a direct consequence by MP of β_h and β_j ($h<i$, $j<i$, β_h is $\beta_j \supset \beta_i$). By induction hypothesis

$$\Gamma \vdash_S C_h^{(1)}\alpha \supset (C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset \beta_h)\ldots) \quad \text{and}$$

$$\Gamma \vdash_S C_j^{(1)}\alpha \supset (C_j^{(2)}\alpha \supset \ldots \supset (C_j^{(p_j)}\alpha \supset \beta_j)\ldots).$$

From these, using P1 and P3, we obtain:

$$\Gamma \vdash_S C_h^{(1)}\alpha \supset (C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset (C_j^{(1)}\alpha \supset \ldots \supset (C_j^{(p_j)}\alpha \supset \beta_h)\ldots),$$

$$\Gamma \vdash_S C_h^{(1)}\alpha \supset (C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset (C_j^{(1)}\alpha \supset \ldots \supset (C_j^{(p_j)}\alpha \supset \beta_j)\ldots),$$

since β_h is $\beta_j \supset \beta_i$, an application of P4 gives (2).

(iii) β_i is a direct consequence by the monotonic rule φ_{k_i} of a wf β_h which depends upon α. ($h<i$, β_h has the form $\gamma \supset \delta$.) By induction hypothesis

$$\Gamma \vdash_S C_h^{(1)}\alpha \supset (C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset \beta_h)\ldots).$$

An application of φ_{k_i} gives:

$$\text{(3)} \qquad \Gamma \vdash_S C_{k_i} C_h^{(1)}\alpha \supset C_{k_i}[\, C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset \beta_h)\ldots)]$$

from which, by iterated use of (V), P1 and P4,

$$\Gamma \ {}_S\ C_{k_i} C_h^{(1)}\alpha \supset (C_{k_i} C_h^{(2)}\alpha \supset \ldots \supset (C_{k_i} C_h^{(p_h)}\alpha \supset (C_{k_i}\gamma \supset C_{k_i}\delta))\ldots).$$

But $C_{k_i}\gamma \supset C_{k_i}\delta$ is just β_i and every $C_{k_i} C_h^{(t)}$ is obtained from the string

$C_{k_i}C_{k_i-1}\cdots C_1$ as required. Therefore (2) holds.

(iv) β_i is a direct consequence by the reinforcing rule φ_{k_i} of a wf $\beta_h (h<i)$ which depends upon α. As in case (iii), the inductive hypothesis and an application of φ_{k_i} give:

$$\Gamma \vdash_S C_{k_i}[C_h^{(1)}\alpha \supset (C_h^{(2)}\alpha \supset \ldots \supset (C_h^{(p_h)}\alpha \supset \beta_h)\ldots)] \ .$$

By (V) and MP we obtain (4), and using as above (V), P1 and P4,

$$\Gamma \vdash_S C_{k_i}C_h^{(1)}\alpha \supset (C_{k_i}C_h^{(2)}\alpha \supset \ldots \supset (C_{k_i}C_h^{(p_h)}\alpha \supset C_{k_i}\ \beta_h)\ldots).$$

But $C_{k_i}\beta_h$ is β_i and every $C_{k_i}C_h^{(t)}$ is obtained from the string $C_{k_i}C_{k_i-1}\cdots C_1$ as required. So (2) follows.

(v) β_i is a direct consequence, by a monotonic or reinforcing rule, of a wf which does not depend upon α. It is easy to prove a lemma analogous to Proposition 2.3 in [4], p.60. Therefore $\Gamma \vdash_S \beta_i$.

Remark. Given a particular deduction $\beta_1,\ldots,\beta_n$ of β from Γ,α, it is always possible to determine effectively how many and which are the wfs $C^{(t)}\alpha$ occurring in the corresponding instance of (1). This is a direct consequence of the constructiveness of our proof.

Now suppose that the system S satisfies a further condition:

VI) For every $\varphi \in R_S$, $\varphi \neq$ MP, and for every wf α of S, $\vdash_S C_\varphi \alpha \supset \alpha$.

Theorem 2. (General Weakened Deduction Theorem). Let S be a system satisfying conditions I/VI. If $\Gamma,\alpha \vdash_S \beta$, then $\Gamma \vdash_S C_k C_{k-1}\cdots C_1 \alpha \supset \beta$.

Proof. We only sketch the proof which is analogous to that of Theorem 1.

This time we show by induction that

(4) $$\Gamma \vdash_S C_{k_i} C_{k_i - 1} \ldots C_1 \alpha \supset \beta_i$$

In the cases (i) and (v) of Theorem 1, (4) follows easily from the observations already done, with the aid of P5 and the new condition VI.

In the case corresponding to (ii), the inductive hypotheses are now

$$\Gamma \vdash_S C_{k_h} C_{k_h - 1} \ldots C_1 \alpha \supset \beta_h \quad \text{and}$$

$$\Gamma \vdash_S C_{k_j} C_{k_j - 1} \ldots C_1 \alpha \supset \beta_j .$$

Since $k_h \leqslant k_i$ and $k_j \leqslant k_i$, by virtue of P5, VI and P4, we obtain (4).

Finally, in the cases (iii) and (iv) of Theorem 1, the induction hypothesis is

$$\Gamma \vdash_S C_{k_h} C_{k_h - 1} \ldots C_1 \alpha \supset \beta_h .$$

Since $k_h < k_i$ (β_h depends upon α), we can write, as above,

$$\Gamma \vdash_S C_{k_i - 1} C_{k_i - 2} \ldots C_1 \alpha \supset \beta_h .$$

and (4) can be easily obtained.

4. We list here some corollaries of our theorems.

The symbols are taken from [5] for the systems CPC, CPI, T, S4, S5, LPC, LPC+T, LPC+S4, LPC+S5, from [6] for Lemmon's K and from [7] for Lemmon's E2.

Corollary 1.1. S is K.

If $\Gamma, \alpha \vdash_S \beta$ and $m \geqslant 0$ is the number of times the rule N of necessitation is applied in the deduction to wfs depending upon α, then

$$\Gamma \vdash_S L^{t_1} \alpha \supset (L^{t_2} \alpha \supset \ldots \supset (L^{t_p} \alpha \supset \beta) \ldots)$$

where $0 \leqslant t_i \leqslant m$ for every $i(1 \leqslant i \leqslant p)$.

Corollary 2.1. S is CPC or CPI.

If $\Gamma,\alpha \vdash_S \beta$ then $\Gamma \vdash_S \alpha \supset \beta$.

Corollary 2.2. S is T (E2).

If $\Gamma,\alpha \vdash_S \beta$ and $m \geqslant 0$ is the number of times the rule N (RM) is applied in the deduction to wfs depending upon α, then $\Gamma \vdash_S L^m\alpha \supset \beta$.

Corollary 2.3. S is S4 or S5 and the other hypotheses are like the case S = T of Corollary 2.2. Then

$$\Gamma \vdash_S L\alpha \supset \beta \quad \text{if} \quad m>0 \quad \text{and} \quad \Gamma \vdash_S \alpha \supset \beta \quad \text{if} \quad m = 0.$$

Corollary 2.4. S is LPC.

If $\Gamma,\alpha \vdash_S \beta$ then $\Gamma \vdash_S \forall x_1 \forall x_2 \ldots \forall x_k \alpha \supset \beta$ where for each $i=1,2,\ldots,k$, the variable x_i occurs quantified in some application of the rule GEN to some wf depending upon α. Moreover, every prefix $\forall x_i$ corresponding to a variable x_i not free in α can be erased. If, in particular, no application of GEN uses a variable free in α, then $\Gamma \vdash_S \alpha \supset \beta$.

Corollary 2.5. S is LPC + T.

If $\Gamma,\alpha \vdash_S \beta$ then $\Gamma \vdash_S C_1 C_2 \ldots C_k \alpha \supset \beta$ where each $C_i (i=1,2,\ldots,k)$ has the form L or $\forall x_j$. The same restrictions of Corollary 2.4 apply. If, in particular, no application of GEN has, as its quantified variable, a free variable of α, then $\Gamma \vdash_S L^m \alpha \supset \beta$, where $m \leqslant k$ is the number of times the rule N is applied to wfs depending upon α.

Corollary 2.6. S is LPC + S4 or LPC + S5.

As in Corollary 2.5, we have in the general case $\Gamma \vdash_S C_1 C_2 \ldots C_k \alpha \supset \beta$ with obvious simplifications among prefixes. In particular:

$$\Gamma \vdash_S L\alpha \supset \beta \quad \text{or} \quad \Gamma \vdash_S \alpha \supset \beta \ .$$

5. In any Hilbert-type system the inference rules can be viewed either as theoremhood rules - say T-rules, - or as deducibility rules - say D-rules. (See, for example, Smiley [8], [9]). Modal rules, for example, are generally considered T-rules (see, for the rule N, [5]), while Modus Ponens in CPC is always treated as a D-rule. According to some authors, even the rule GEN of the Classical Predicate Calculus has to be used as a T-rule ([8]). Elsewhere it is given as a D-rule (e.g. in [4]). More often this distinction is undervalued or simply ignored.

In our opinion, it is useful to differentiate two kinds of deductions: a free deduction, the one adopted in this work, modelled as in [4], and a conditioned deduction, in all analogous to the previous one, except for some restrictions on the use of at least one of the inference rules (for a typical example relative to the rule GEN, see [10], p.112 and [11], p. 31). A stimulating discussion about this problem can be found in an interesting paper of Henkin and Montague [12].

Till now all concerns the syntax of a system. The semantical counterpart of the syntactical notion of deducibility, at least for those systems which admit a tarskian semantics, is the classical notion of logical consequence.

As usual, the expression '$\alpha \models \beta$' means that β is a logical consequence of α. It is well-known that if $\alpha \models \beta$ then $\models \alpha \supset \beta$, i.e. the wf $\alpha \supset \beta$ is logically valid.

The corresponding syntactical property is the Deduction Theorem. A sound notion of deduction must guarantee that if $\alpha \vdash \beta$, then $\vdash \alpha \supset \beta$. The free deduction does not meet this request in all cases. For example, in the classical first-order predicate calculus the Deduction Theorem suffers from some limitations (see [4], p.61), due to choice of a free notion of deduction. For this reason some authors prefer to consider GEN as a theoremhood rule, as far as GEN preserves the truth (relative to a given interpretation) but not always the satisfiability. On the contrary, a standard Deduction Theorem can be derived for the classical predicate calculus introducing a suitable conditioned deduction, as in Kleene [10].

An analogous problem arises in connection with the modal rule N. If the rule is used as a D-rule - as in the present paper -, the following happens:

(i) $\frac{\alpha}{L\alpha}$ preserves the truth in a model.

Obviously, the validity is also preserved.

(ii) $\frac{\alpha}{L\alpha}$ does not preserve the truth in a single world.

Unfortunately for modal systems no natural syntactical restriction seems available for a notion of deduction.

A proposal could be to extend whenever possible T-rules to D-rules, renouncing to a standard deduction theorem in favour of a weak version of it. This is supported by two arguments: the notion of free deduction is more general and natural (in the sense of less ad hoc), and the notion of T-rule can be objected both from a philosophical and a formal point of view (as Hacking [13], for example, does).

Unfortunately, the above proposal can hardly be carried out - as argued in [14] - for Lukasiewicz many-valued logical systems.

References

[1] ZEMAN, J.J. Modal Logic. Oxford at the Clarendon Press, 1973.

[2] PERZANOWSKI, J. "The Deduction Theorems for the Modal Propositional Calculi formalized after the manner of Lemmon". Reports on Mathematical Logic, 1 (1973).

[3] PERZANOWSKI, J. "On M-fragments and L-fragments of Normal Modal Propositional Logics". Reports on Mathematical Logic, 5 (1975).

[4] MENDELSON, E. Introduction to Mathematical Logic. Van Nostrand, 1964.

[5] HUGHES, G.E. and CRESSWELL, M.J. An Introduction to Modal Logic. Methuen & Co., 1968.

[6] LEMMON, E.J. An Introduction to Modal Logic. American Philosophical Quarterly (Monograph Series n°. 11), 1977.

[7] LEMMON, E.J. "New Foundations for Lewis Modal Systems". J.S.L., 22 (1957).

[8] SMILEY, T. J. "Relative Necessity" J.S.L., 28 (1963).

[9] SMILEY, T. J. "The Logical Basis of Ethics". Acta Philosophica Fennica, 16, (1963).

[10] KLEENE, S.C. Mathematical Logic John Wiley & Sons, 1967.

[11] BARNES, D. W. and MACK, J. M. An Algebraic Introduction to Mathematical Logic. Springer Verlag, 1975.

[12] HENKIN, L. and MONTAGUE, R. "On the definition of 'Formal Deduction' ". J.S.L., 21, (1956).

[13] HACKING, I. "What is Logic?". The Journal of Philosophy, 76, (1979).

[14] GUCCIONE, S., TORTORA, R. and VACCARO, V. "Deduction Theorems in Lukasiewicz Propositional Calculi". Prepint.

(*) Istituto di Fisica Teorica
Università di Napoli.

(**) Istituto di Matematica
Università di Napoli.