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# A NOTE ON THE p-DISTRIBUTIVITY IN NON-ARCHIMEDEAN f-RINGS

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*ABSTRACT*

*Non-archimedean f-rings need not be p-distributive. Moreover, if $\{d_i | i\}$ is a subset of a non-archimedean f-ring and $a \geqslant 0$, the elements $a \bigvee_i d_i$ and $\bigvee_i ad_i$ need not be equal. We prove, however, that the difference is an infinitely small element when the ring has a strong unity.*

Recall first from [5] that a lattice-ordered ring A is <u>left p-distributive</u> (respectively <u>right p-distributive</u>) if whenever $\bigvee_i d_i$ exists (with $\{d_i | i\} \subset A$) and $a \geqslant 0$, then $\bigvee_i ad_i$ (resp. $\bigvee_i d_i a$) also exists and $a \bigvee_i d_i = \bigvee_i ad_i$ (resp. $(\bigvee_i d_i)a = \bigvee_i (d_i a)$). A is <u>p-distributive</u> if it is left and right p-distributive

Archimedean f-rings are p-distributive [3]. With independence of the hypothesis of archimedeanity it is possible, however, to find p-distributive f-rings: a) Commutative f-rings with unity, in which every non-unit is a zero-divisor, are p-distributive [5]. b) Bounded inversion f-rings are also p-distributive, as is shown immediately from propositions 1 and 2 and lemma 1 of [5]. We recall that an f-ring is of bounded inversion if every $x \geqslant 1$ is a unit.

It may be asked whether there exist non p-distributive non-archimedean f-rings or not. Some examples related with the question follow:

Example 1. Let X be a non-pseudocompact topological space and let C(X) be the f-ring of real continuous functions defined on X, under pointwise ordering and operations. Let M be a hyper-real maximal ideal [2] and A=C(X)/M the canonically ordered quotient ring. Then A is a totally ordered non-archimedean field. Since every $x > 0$ is a unit, A is p-distributive [5].

Example 2. Let R[x] be the ring of polinomials in an indeterminate x with real coefficients, endowed with the usual operations and the total ordering defined as follows: if $P(x)=a_n x^n + \ldots + a_o$ $(a_n \neq 0)$, then $P > 0$ if and only if $a_n > 0$. We thus obtain a non-archimedean f-ring that is not p-distributive, since for example $x(\bigwedge_{n=1}^{\infty} n^{-1}) = 0$ and the infimum of $\{xn^{-1} \mid n \in \mathbb{N}\}$ does not exist.

In the non p-distributive case, a natural question arises: which is the relation, if there is any, between $a \bigvee_i d_i$ and $\bigvee_i ad_i$, if we suppose that both suprema exist?.

Before giving an answer to this question, recall that an element $x \in A$ is infinitely small with respect to $y \in A$ whenever $n|x| \leqslant |y|$ holds for every $n \in N$ [4] ($x \ll y$, for short). If $I_0(y) = \{x \in A \mid x \ll y\}$, we write $I_0(A) = \bigcup_{y \in A} I_0(y)$. Also, an element is said to be a strong unity if it is contained in no proper solid subgroup [1].

Definition. The elements x,y of a lattice-ordered ring A are called infinitely close if $x-y \in I_0(A)$.

We now state the main result of this note:

Theorem. Let A be a non archimedean lattice-ordered ring with a strong unity u. Then:

a) $I_0(A)$ is a closed solid ideal.

b) If A is besides an f-ring, $a\geqslant 0$, and $\{d_i | i\}$ is a subset of A such that $\bigvee_i d_i$, $\bigvee_i ad_i$ (respectively $\bigvee_i d_i a$) exist, then $a \bigvee_i d_i$ and $\bigvee_i ad_i$ are <u>infinitely close</u> (respectively, so are $(\bigvee_i d_i)a$ and $\bigvee_i (d_i a)$).

<u>Proof.</u> Note first that $I_0(A) \subset I_0(u)$. Indeed, if $z \ll g$ for some $g\geqslant 0$, there exists $n_1 \in N$ such that $n_1 n|z| \leqslant n_1|u|$ holds for every $n\in N$. So $n|z| \leqslant |u|$ for every $n\in N$. Now,

a) It is clear that $I_0(A)$ is a solid ideal (in the ring-theoretic sense); let now $\{x_j|j\}$ be a subset of $I_0(A)$ such that $x = \bigvee_j x_j$ exists in A. We must prove that $x\in I_0(A)$. By the preceding remark, we have $nx_j^+ \leqslant |u|$ and $nx_j^- \leqslant |u|$, $\forall n\in N$, $\forall j$. Hence, using $x^+ = \bigvee_j x_j^+$ and $x^- = \bigwedge_j x_j^-$, we obtain $x\in I_0(A)$.

b) Since $I_0(A)$ is closed (a)), the canonical mapping of A onto $A/I_0(A)$ preserves the suprema of subsets of A [4]. By the remark above, $I_0(A/I_0(A))=0$, and so $A/I_0(A)$ is an archimedean f-ring. Hence it is p-distributive, and this completes the proof.

## <u>References</u>

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