STOCHASTICA, Vol. IV, nº 1 (1980)

ON THE CENTRAL LIMIT THEOREM IN HILBERT SPACE(*)

Evarist Giné(1) and José R. León(2)

1. Introduction

The object of this paper is to prove a central limit theorem in (separable) Hilbert space using a method based on the so called "découpage de Lévy", the Lindeberg proof for the Gaussian case and an elementary proof of Poisson convergence for the direct part, and on elementary probabilistic inequalities for the converse. In particular, characteristic functions are only used in unicity questions. Several results of Varadhan (1962) can be obtained either directly as corollaries of the main theorem or with the same methods.

This is essentially an expository article: many of the results in it are formally new but perfectly predictable given the previous work of Varadhan (1962) and Le Cam (1970). We hope the reader will find this approach to the central limit theorem both more clarifying and less computationally involved than the usual ones.

The direct part of the central limit theorem is proved as follows. By means of a randomization -the découpage de Lévy- the general central limit theorem is reduced to the cases of normal convergence and Poisson convergence: in fact, the laws of usually small variables -variables in an infinitesimal array- are approximately decomposed into convolutions of laws of small variables and laws of variables usually zero. Then, sums of small variables are approximated

(*) This author's research has been supported by a scholarship from the Fundación Gran Mariscal de Ayacucho.

by Gaussian laws (this is done using a generalization of the Lindeberg method in Hilbert space) and sums of variables usually zero are approximated by generalized Poisson probabilities (and this can be achieved by means of an elementary argument on approximation of exponentials). As a result, one obtains that sums of usually small variables are well approximated by convolutions of Gaussian and generalized Poisson laws.

The converse central limit theorem consists essentially in showing that if $\{L(S_n)\}$ is tight, where S_n is the n-th row sum in an infinitesimal array, then the measures

$$\Sigma_j [\chi_{\{\|x\|\leqslant 1\}}(x)\|x-\int_{\|x\|\leqslant 1} x\,dL(X_{nj})(x)\|^2 dL(X_{nj})(x)+\chi_{\{\|x\|>1\}}(x)\,dL(X_{nj})(x)]$$

are also tight. Again by the découpe de Lévy the proof of this fact can be easily achieved by means of the classical Lévy and converse Kolmogorov's inequalities. For generalizations to certain Banach spaces we refer to de Acosta, Araujo and Giné (to appear), and for another approach to the subject, to the already cited paper of Varadhan (1962) (or Parthasarthy (1967)).

We are grateful to Professor L. LeCam from whom we learned this approach to the central limit theorem. We are also indebted to A. de Acosta and D. Freedman for several useful conversations.

In what follows H is always a <u>separable</u> Hilbert space.

2. Differentiable functions and weak convergences

We recall that if $\mu_n, \mu \in P(H)$, $\mu_n \to_w \mu$ or $w\text{-}\lim_{n\to\infty}\mu_n=\mu$ means that $\int f d\mu_n \to \int f d\mu$ for every bounded continuous function $f:H\to R$. We prove here that in order to check weak convergence of $\{\mu_n\}$ to μ it is enough to show that $\int f d\mu_n \to \int f d\mu$ for every $f:H\to R$ continuous, bounded and with bounded derivatives of all orders, just as in the real line case.

Let $\phi_\varepsilon : R \to R$ be defined as

$$\phi_\varepsilon(t) = \chi_{(-\infty,1]}(t) + c^{-1}\left[\int_0^{1-(t-1)(2\varepsilon+\varepsilon^2)^{-1}} \exp\{-s^{-1}(s-1)^{-1}\}ds\right]\chi_{[1,(1+\varepsilon)^2]}(t)$$

where $c = \int_0^1 \exp\{-s^{-1}(s-1)^{-1}\}ds$. Define now

$$\psi_\varepsilon(x) = \phi_\varepsilon(\|x\|^2).$$

Then, since the derivatives of $\|x\|^2$ are

$$(\|x\|^2)'(h) = 2<x,h>$$

$$(\|x\|^2)''(h,k) = 2<h,k>$$

$$(\|x\|^2)^{(r)}(h_1,\dots,h_r) \equiv 0 \quad \text{for } r>2,$$

it is a matter of simple routine computation to show that for every r, $\psi_\varepsilon^{(r)}(x)(h_1,\dots,h_r)$ is a linear combination of numbers of the form

$$\phi_\varepsilon^{(h)}(\|x\|^2)<x,h_{\tau(1)}>\dots<x,h_{\tau(k)}><h_{\tau(k+1)},h_{\tau(k+2)}>\dots<h_{\tau(r-1)},h_{\tau(r)}>.$$

(For definition and properties of differentiable functions in infinite dimensional spaces, see e.g. Lang (1962)).

As a consequence, since $\sup_{t\in R}|\phi_\varepsilon^{(k)}(t)|<\infty$, and $\phi_\varepsilon^{(k)}$ has a compact support for $k\neq 0$, there exists $C_k \geqslant 0$ such that for every $x\in H$ and $(h_1,\dots,h_n)\in H^n$,

$$|\psi_\varepsilon^{(k)}(x)(h_1,\dots,h_k)| \leqslant C_k\|h_1\|\dots\|h_k\|.$$

The infimum of the numbers C_k will be denoted by $\|\psi_\varepsilon^{(k)}\|_\infty$. So:

2.1. Lemma. For every $\varepsilon>0$ and $k=0,1,\ldots,$ the function ψ_ε satisfies $\|\psi_\varepsilon^{(k)}\|_\infty<\infty$, i.e. ψ_ε is bounded and has bounded derivatives of all orders.

Denote by $C_b^\infty(H)$ the class of bounded functions with bounded derivatives.

2.2. Theorem. Let $\mu_n,\mu\in P(H)$. Then if $\int fd\mu_n\to\int fd\mu$ for every function $f\in C_b^\infty(H)$, $\mu_n\to_w\mu$.

Proof. It is enough to see that if $\int fd\mu_n\to\int fd\mu$ for every $f\in C_b^\infty(H)$, then $\mu_n(F)\to\mu(F)$ for every finite intersection of balls F of μ-continuity (see e.g. Billingsley (1968) page 14). Let then $F=\cap_{i=1}^r B(x_i,r_i)$, where $B(x_i,r_i)=\{x:\|x-x_i\|<r_i\}$. For $0<\varepsilon<r_i$, define

$$F^\varepsilon=\cap_{i=1}^n B(x_i,r_i+\varepsilon),\ F_\varepsilon=\cap_{i=1}^n B(x_i,r_i-\varepsilon),\ \tau^\varepsilon(x)=\Pi_{i=1}^n\psi_{\varepsilon r_i^{-1}}(r_i^{-1}(x-x_i))$$

and $\tau_\varepsilon(x)=\Pi_{i=1}^n\psi_{\varepsilon(r_i-\varepsilon)^{-1}}((r_i-\varepsilon)^{-1}(x-x_i))$. Then, $\tau_\varepsilon,\tau^\varepsilon\in C_b^\infty$ and

$\chi_{F_\varepsilon}\leqslant\tau_\varepsilon\leqslant\chi_F\leqslant\tau^\varepsilon\leqslant\chi_{F^\varepsilon}$ (where for every set A, χ_A is the indicator function of A). So, if F is a continuity set for μ,

$$\limsup_{n\to\infty}\mu_n(F)\leqslant\lim_{\varepsilon\to0}\lim_{n\to\infty}\int\tau^\varepsilon d\mu_n=\lim_{\varepsilon\downarrow0}\int\tau^\varepsilon d\mu=\mu(F)$$

$$=\lim_{\varepsilon\downarrow0}\int\tau_\varepsilon d\mu=\lim_{\varepsilon\downarrow0}\lim_{n\to\infty}\int\tau_\varepsilon d\mu_n\geqslant\liminf_{n\to\infty}\mu_n(F).$$

Hence, $\lim_{n\to\infty}\mu_n(F)=\mu(F)$.

2.3. Definition. For every positive integer k, d_k is the distance on $P(H)$ given by

$$d_k(\mu,\nu)=\sup\{|\int fd(\mu-\nu)|:f\in C_b^k(H),\Sigma_{r=0}^k\|f^{(r)}\|_\infty\leqslant1\},$$

where $C_b^k(H)$ denotes the set of bounded functions with k bounded continuous derivatives.

Certainly, if $d_k(\mu_n,\mu)\to 0$ then $\mu_n \to_w \mu$ by the last theorem. The converse is also true:

2.4. Theorem. For every k>0, the distance d_k metrizes the weak topology of P(H).

Proof. In the first place (P(H), weak topology) is metrizable (see e.g. Dudley (1966)). If $\mu_n \to_w \mu$ then $d_k(\mu_n,\mu)\to 0$ by a re-sult of Ranga Rao (1962). Conversely if $d_k(\mu_n,\mu)\to 0$, then $\mu_n \to_w \mu$ by Theorem 1.2.

Remark. Theorem 2.4 is not needed in what follows.

3. The main lemmas.

In this section, among other needed facts we prove simple Gaussian and Poisson approximation results which will give the direct central limit theorem and a necessary condition for tightness of sums (the analogue of the two-series critèrion of P. Lévy for triangular arrays) which is essentially the converse central limit theorem.

3.1. Lemma. For every k and $\{\mu_i,\nu_i\}_{i=1}^n \epsilon P(H)$,

$$d_k(\mu_1 * \dots * \mu_n, \nu_1 * \dots * \nu_n) \leqslant \Sigma_{i=1}^n d_k(\mu_i,\nu_i).$$

Proof. It follows trivially, by use of Fubini's theorem, from the invariance of the set $\{f:H\to R,\ f\epsilon C_b^k(H),\ \Sigma_{i=0}^k \|f^{(i)}\|_\infty\}$ under transformations of the form $f(.)\to f(x+.)$, $x\epsilon H$.

3.2. Theorem. Let $\{X_i\}_{i=1}^n$ be independent centered H-valued rv's such that $\|X_i(\omega)\| \leqslant C$ for all $\omega\epsilon\Omega$, i=1,...,n and some C>0. Let $S=\Sigma_{i=1}^n X_i$, A_i the covariance operator of X_i and $A=\Sigma_{i=1}^n A_i$ that of S. Then, if N(0,A) is the centered Gaussian p.m. with covariance A,

$$d_3(L(S),\ N(0,A)) \leqslant KCtrA,$$

where K may be taken to be $K=6^{-1}(1+3^{3/4})$.

<u>Proof.</u> Let Y_i be independent $N(0,A_i)$ rv's (they exist as the A_i are trace class -see Lemma 3.5). Then, $L(\Sigma_{i=1}^n Y_i) = N(0,A)$ and by the previous lemma,

$$d_3(L(S),\ L(\Sigma_{i=1}^n Y_i)) \leqslant \Sigma_{i=1}^n d_3(L(X_i),L(Y_i)). \tag{3.1}$$

If $f\epsilon C_a^3(H)$, by Taylor's theorem (see e.g. Lang (1962)),

$$f(X_i)=f(0)+f'(0)(X_i)+\frac{1}{2}f''(0)(X_i,X_i)+6^{-1}f'''(\theta)(X_i,X_i,X_i),$$

and likewise for $f(Y_i)$. So,

$$|E(f(X_i)-f(Y_i))| \leqslant \frac{1}{2}|E\{f''(0)(X_i,X_i)-f''(0)(Y_i,Y_i)\}| \tag{3.2}$$
$$+\ 6^{-1}\|f'''\|_\infty E(\|X_i\|^3+\|Y_i\|^3).$$

Next we estimate the right hand side of this inequality.

Note that

$$E|f''(0)(X_i,X_i)| \leqslant E\|f''(0)\|\|X_i\|^2 = \|f''(0)\|E\|X_i\|^2<\infty,$$

i.e. $f''(0)(X_i,X_i)\epsilon L_1(\Omega)$. For ease of notation let us suppress the subindex of X in what follows. Let now $\{e_i\}$ be a cons for H and let $\xi_i=<X,e_i>$ and $X^n=\Sigma_{i=1}^n \xi_i e_i$.

Then $|f''(0)(X^n,X^n)| \leqslant \|f''(0)\|\|X^n\|^2 \leqslant \|f''(0)\|\|X\|^2$ i.e. the random variables $|f''(0)(X^n,X^n)|$ are dominated by an integrable function. On the other hand,

$$|f''(0)(X,X)-f''(0)(X^n,X^n)|=|f''(0)(X,X-X^n)-f''(0)(X^n-X,X^n)|$$

$$\leqslant \|f''(0)\|\,\|X-X^n\|\,[\,\|X\|+\|X^n\|\,]\to 0,$$

and therefore, by the Lebesgue dominated convergence theorem

$$Ef''(0)(X^n,X^n)\to Ef''(0)(X,X).$$

But $\quad Ef''(0)(X^n,X^n)=\Sigma_{i,j=1}^n f''(0)(e_i,e_j)E\langle X,e_i\rangle\langle X,e_j\rangle$. Hence

$Ef''(0)(X,X)$ depends only on f and the covariance of X. Therefore, since X_i and Y_i have the same covariance,

$$Ef''(0)(X_i,X_i) = Ef''(0)(Y_i,Y_i),$$

and the estimate (3.2) becomes

$$|E(f(X_i) - f(Y_i))| \leqslant 6^{-1}\|f'''\|_\infty E(\|X_i\|^3+\|Y_i\|^3).$$

To estimate this last quantity, let us note that if Y is Gaussian, $Y=\Sigma_{i=1}^\infty \eta_i e_i$, $E\eta_i=0$ and $E\eta_i^2=\sigma_i^2$, then $E\eta_i^4=3\sigma_i^4$ and

$$E\|Y\|^3 = E(\Sigma_{i=1}^\infty \eta_i^2)^{3/2} \leqslant [E(\Sigma\eta_i^2)^2]^{3/4}$$

$$=[\Sigma_{i,j}\sigma_i^2\sigma_j^2 + 3\Sigma_i\sigma_i^4 - \Sigma_i\sigma_i^2\sigma_i^2]^{3/4}$$

$$=[(\Sigma_i\sigma_i^2)^2 + 2\Sigma_i\sigma_i^4]^{3/4}\leqslant 3^{3/4}(\Sigma_i\sigma_i^2)^{3/2}.$$

So, in our case we get $E\|Y_i\|^3 \leqslant 3^{3/4}\mathrm{CtrA}_j$. As for X_i we have $E\|X_i\|^3 \leqslant CE\|X_i\|^2 = \mathrm{CtrA}_i$ and therefore

$$|E(f(X_i)) - f(Y_i))| \leqslant 6^{-1}(1+3^{3/4})\ \mathrm{CtrA}_i.$$

Now the theorem follows from (3.1).

The last theorem makes precise the statement that "laws of sums of small variables are well approximated by Gaussian laws". A similar approach is used in the work of Kuelbs and Kurtz (1974).

Given a finite positive measure μ on H, define

$$\mathrm{Pois}\mu = e^{-|\mu|} \Sigma_{n=0}^{\infty} \mu^n/n! = e^{\mu-|\mu|\delta_0}$$

where $|\mu|=\mu(H)$. Define also

$$\mathrm{cPois}\mu = (\mathrm{Pois}\mu)*\delta_u$$

where $u=-\int \min(1,\|x\|)\|x\|^{-1}x\,d\mu(x)$. (Pois is for <u>Poisson</u> and cPois for <u>centered Poisson</u>).

<u>3.3. Lemma.</u> (i) $\mathrm{Pois}(\mu_1+\ldots+\mu_n)=(\mathrm{Pois}\mu_1)*\ldots*(\mathrm{Pois}\mu_n)$,

(ii) if $\mu(H) \geqslant \nu(H)$, $d_1(\mathrm{Pois}\mu, \mathrm{Pois}\nu) \leqslant d_1(\mu,\nu) + (1-e^{|\nu|-|\mu|})$,

(iii) if $\mu \in P(H)$, $\sup_{A\in\mathcal{B}}|\mu(A)-\mathrm{Pois}\mu(A)| \leqslant e^2[\mu(H\setminus\{0\})]^2$.

<u>Proof.</u> (i) It is enough to prove it for μ and ν.

$$\begin{aligned}\mathrm{Pois}\mu*\mathrm{Pois}\nu &= e^{-|\mu|-|\nu|}\Sigma\mu^n/n! * \Sigma\nu^n/n! \\ &= e^{-|\mu+\nu|}\Sigma_{n,m}\mu^n*\nu^m/n!\,m! \\ &= e^{-|\mu+\nu|}\Sigma_{r=0}^{\infty}\Sigma_{n+m=r}\mu^n*\nu^m/n!\,m! \\ &= e^{-|\mu+\nu|}\Sigma_{r=0}^{\infty}(\mu+\nu)^r/r! = \mathrm{Pois}(\mu+\nu).\end{aligned}$$

(ii) If μ and ν do not have the same mass then it is easy to see that, similarly to Lemma 2.1,

$$d_k(\mu^n,\nu^n) \leq \max(|\mu|,|\nu|)^{n-1} n d_k(\mu,\nu).$$

Then, if $\Sigma_{r=0}^{k}\|f^{(k)}\|_\infty \leq 1$,

$$|\int fd(\text{Pois}\mu-\text{Pois}\nu)| = |e^{-|\mu|}\Sigma\int fd\mu^n/n! - e^{-|\nu|}\Sigma\int fd\nu^n/n!|$$

$$\leq e^{-|\mu|}|\Sigma\int fd(\mu^n-\nu^n)/n!| + |(e^{-|\mu|}-e^{-|\nu|})\Sigma\int fd\nu^n/n!|$$

$$\leq e^{-|\mu|}e^{\max(|\mu|,|\nu|)}d_k(\mu,\nu) + [1-e^{|\nu|-|\mu|}].$$

(iii) Let $\|.\|$ be the total variation norm; by its well known properties, we have

$$\sup_{A\in\mathcal{B}}|(\mu-\text{Pois}\mu)(A)| = \frac{1}{2}\|\mu-\text{Pois}\mu\| = \frac{1}{2}\|\mu-e^{\mu-\delta_0}\|$$

$$= \frac{1}{2}\|\Sigma_{n=2}^{\infty}(\mu-\delta_0)^n/n!\| \leq 4^{-1}\|\mu-\delta_0\|^2\Sigma_{n=2}^{\infty}2^n/(n-2)! = 4^{-1}e^2\|\mu-\delta_0\|^2$$

$$= e^2[\mu(H\backslash\{0\})]^2. \quad \square$$

<u>Remark.</u> e^2 is not the best constant in (iii). With a slightly more conplicated proof one can obtain $\sup_{A\in\mathcal{B}}|\mu(A)-\text{Pois}\mu(A)| \leq [\mu(H\backslash\{0\})]^2$ (see e.g. Freedman (1974)).

It is easy to see that as in Lemma 3.1,

$$\sup_{A\in\mathcal{B}}|\mu_1*\ldots*\mu_n(A)-\nu_1*\ldots*\nu_n(A)|$$

$$\leq \Sigma_{i=1}^{n}\sup_{A\in\mathcal{B}}|\mu_i(A)-\nu_i(A)|.$$

So, the last lemma shows that

<u>3.4. Theorem.</u> If $\{\mu_i\}_{i=1}^{n}\in P(H)$, then

$$\sup_{A\in\mathcal{B}}|(\mu_1*\ldots*\mu_n-\text{Pois}\Sigma_{i=1}^{n}\mu_i)(A)| \leq e^2\Sigma_{i=1}^{n}[\mu_i(H\backslash\{0\})]^2.$$

The last theorem shows that the laws of sums of "variables usually zero are well approximated by Poisson p.m.'s".

Next we give some facts on Gaussian and Poisson laws. The next well known lemma has been implicitely used in Theorem 3.2. We give here a simple proof which does not use characteristic functions.

3.5. Lemma. A positive definite bilinear form Φ on H is the covariance of a Gaussian p.m. on H if and only if there exists a positive Hermitian trace class (nuclear) operator A such that $\Phi(x,x)=\langle Ax,x\rangle$ for every $x\in H$.

Proof. Let A be trace class positive Hermitian, $\{e_i\}$ a cons of eigenvectors of A, $\{\lambda_i\}$ the corresponding eigenvalues, $\lambda_i>0$, $\Sigma\lambda_i<\infty$, $\{\eta_i\}$ a sequence of N(0,1) independent real rv's and $X=\Sigma\lambda_i\eta_i e_i$. This series is Cauchy in probability as $E\|\Sigma_{i=N}^{\infty}\lambda_i\eta_i e_i\|^2 = =\Sigma_{i=N}^{\infty}\lambda_i\to 0$ and X is well defined. Clearly X is Gaussian and $E\langle X,y\rangle^2 = \langle Ay,y\rangle$.

Conversely, if X is Gaussian, Φ its covariance and $\langle Ax,x\rangle= =\Phi(x,x)$ then, since $E\|X\|^2<\infty$ by Fernique's theorem (Fernique (1970)), A is trace class: $\Sigma\langle Ae_i,e_i\rangle=\Sigma E\langle X,e_i\rangle^2 = E\|X\|^2<\infty$.

3.6. Lemma. If μ is a positive measure on H such that $\int\min(1,\|x\|^2)d\mu(x)<\infty$, then, $\lim_{\delta\downarrow 0}c\mathrm{Pois}(\mu|\|x\|>\delta)$ exists. Conversely, if this limit exists then $\int\min(1,\|x\|^2)d\mu(x)<\infty$.

For the proof see Araujo and Giné (to appear) or Parthasarathy (1967). We only use the direct part in what follows.

3.7. Definition. A positive measure μ on H is a Lévy measure if $\int\min(1,\|x\|^2)d\mu(x)<\infty$. In this case, $\lim_{\delta\downarrow 0}c\mathrm{Pois}(\mu|\|x\|>\delta)$ is called the centered Poisson p.m. with Lévy measure μ, cPoisμ for short.

<u>3.8. Definition.</u> A family $\{X_{nj}:j=1,\dots,k_n, n=1,\dots\}$ of random variables is an <u>infinitesimal system</u> if for each n, $X_{n1},\dots,X_{nk_n}$ are independent, and for each $\varepsilon<0$, $\lim_{n\to\infty}\sup_j P\{\|X_{nj}\|>\varepsilon\}=0$.

Next we will give a necessary condition for tightness of partial sums of an infinitesimal system which essentially contains the converse central limit theorem.

We recall that the P. Lévy and converse Kolmogorov inequalities are true in Hilbert space, and in general, in any separable Banach space (Parthasarathy (1967), Kahane (1968), de Acosta and Samur (to appear)). These inequalities combined give the following result. For the first part of the proof we follow de Acosta and Samur (to appear), Theorem 2.3.

<u>3.9. Theorem.</u> Let $\{X_{nk}\}$, be an infinitesimal system such that $\{L(S_n)\}_{n=1}^{\infty}$ is relatively shift compact. Then, the family of finite measures

$$d\nu_n^{\delta}(\omega) = \begin{cases} \delta^{-2}\sum_{j=1}^{k_n}\|x-\int_{\|X_{nj}\|\leq\delta} X_{nj}\,dP\|^2 dL(X_{nj})(x) & \text{for } \|x\|\leq\delta, \\ \sum_{j=1}^{k_n} dL(X_{nj})(x) & \text{for } \|x\|>\delta \end{cases}$$

is relatively compact for every $\delta>0$.

<u>Proof.</u> We need only prove the theorem for some $\delta>0$. Let $\tilde{X}_{nj}$ denote independent symmetrizations of the X_{nj} and $\tilde{S}_n=\sum_{j=1}^{k_n}\tilde{X}_{nj}$. Then one of Lévy's inequalities (de Acosta and Samur (to appear)) states that for K compact convex and symmetric,

$$P\{\tilde{X}_{nj}\in K^c \text{ for some } j\} \leq 2P\{\tilde{S}_n\in K^c\}. \tag{3.3}$$

Then,

$$1-2P\{\tilde{S}_n \in K^c\} \leq \Pi_{j=1}^{k_n} P\{\tilde{X}_{nj} \in K\} \leq \Pi_{j=1}^{k_n} e^{-P\{\tilde{X}_{nj} \in K^c\}}$$

or

$$(3.4) \qquad \Sigma_{j=1}^{k_n} P\{\tilde{X}_{nj} \in K^c\} \leq -\log(1-2P\{\tilde{S}_n \in K^c\}).$$

Now, given $\varepsilon>0$ there exists D_ε compact convex and symmetric such that

$$P\{\tilde{S}_n \in D_\varepsilon{}^c\} < \tfrac{1}{2}[1-\exp(-\varepsilon/2)].$$

Since $\{L(X_{nj}):j=1,\ldots,k_n,\ n=1,\ldots\}$ is relatively compact by infinitesimality, by Prokhorov's theorem there exists C compact convex symmetric suth that $P\{X_{nj} \in C^c\} < \frac{1}{2}$ for all n,j. If $K_\varepsilon = D_\varepsilon + C$ and X'_{nj} is independent of X_{nj} and has the same law, we obtain

$$\tfrac{1}{2}P\{X_{nj} \in K_\varepsilon^c\} \leq P\{X'_{nj} \in C,\ X_{nj} \in K_\varepsilon{}^c\} \leq P\{\tilde{X}_{nj} \in D_\varepsilon{}^c\}.$$

Therefore, this inequality together with (3.4) gives:

$$\Sigma_{j=1}^{k_n} P\{X_{nj} \in K_\varepsilon{}^c\} \leq 2\Sigma_{j=1}^{k_n} P\{X_{nj} \in D_\varepsilon{}^c\}$$

$$\leq -2[\log(1-(1-\exp-\varepsilon/2))] = \varepsilon.$$

In particular, $\Sigma_{j=1}^{k_n} P\{X_{nj} \in K_1{}^c\} < \infty$ and $\Sigma_{j=1}^{k_n} P\{X_{nj} \in K_1{}^c \cap K_\varepsilon{}^c\} < \varepsilon$, i.e. $\{\Sigma_{j=1}^{k_n} L(X_{nj})\,|\,\|x\| > \delta\}_{n=1}^\infty$ is relatively compact for every $\delta > \sup\{\|x\| : x \in K_1\}$.

Next we treat the origin. Define .

$$X_{nj\delta}(\omega) = \begin{cases} X_{nj}(\omega) & \text{if} \quad \|X_{nj}(\omega)\| \leq \delta \\ 0 & \text{otherwise,} \end{cases}$$

and let $\tilde{X}_{nj\delta}$ be independent symmetrizations of the $X_{nj\delta}$. Then, by the Lévy and converse Kolmogorov inequalities

$$\Sigma_{j=1}^{k_n} E\| X_{nj\delta}-EX_{nj\delta}\|^2=2^{-1}\Sigma_{j=1}^{k_n} E\|\tilde{X}_{nj\delta}\|^2$$
$$\leqslant [(t+\delta)^2+t^2/2]/[1-4P\{\|\Sigma_{j=1}^{k_n}\tilde{X}_{nj\delta}\|>t\}].$$

So, by the first part of the proof, the theorem will be proved if there exists $t>0$ and $\delta>\sup\{\|x\|:x\epsilon K_1\}$ such that

(3.5) $$P\{\| \Sigma_{j=1}^{k_n}\tilde{X}_{nj\delta}\|>t\}<1/4$$

for every n.

A way of proving (3.5) is to use a result on tightness of convolution factors (Parthasarathy (1967) III.2.2) and the <u>découpage de Lévy</u> (Le Cam (1970)), which we describe immediately.

Let $U_{nj\delta}, V_{nj\delta}, \xi_{nj\delta}$ and $\eta_{nj\delta}$, $j=1,\ldots,k_n$, be independent random variables with laws

$$L(U_{nj\delta}) = L(X_{nj}\,|\,\|x\| \leqslant \delta)$$

$$L(V_{nj\delta}) = L(X_{nj}\,|\,\|x\| > \delta)$$

$$L(\xi_{nj\delta}) = L(\eta_{nj\delta}) = \text{Bernoulli with expectation}$$

$$P\{\|X_{nj}\|>\delta\}$$

where for every Borel set $A, L(X_{nj}\,|\,\|x\| \leqslant \delta)(A)=P[A\cap\{\|X_{nj}\|\leqslant\delta\}]/P\{\|X_{nj}\|\leqslant\delta\}$ if $P\{\|X_{nj}\| \leqslant\delta\}\neq 0$ and $L(X_{nj}|\,\|x\|\leqslant \delta)(A)=\delta_0(A)$ otherwise, and likewise for $L(X_{nj}\,|\,\|x\|>\delta)$. Then it is clear that

$$L(X_{nj\delta})=L(\eta_{nj\delta}U_{nj\delta}),\; L(X_{nj}-X_{nj\delta})=L((1-\xi_{nj\delta})V_{nj\delta})$$

and

(3.6) $$L(X_{nj})=L[\eta_{nj\delta}U_{nj\delta}+(1-\xi_{nj\delta})V_{nj\delta}+(\xi_{nj\delta}-\eta_{nj\delta})U_{nj\delta}].$$

This is the <u>découpage de Lévy</u> of the law of X_{nj}. Note that the first two summands at the right hand side are independent. Next we prove that the third is negligible. We have

(3.7) $$E\|\Sigma_{j=1}^{k_n}(\xi_{nj\delta}-\eta_{nj\delta})U_{nj\delta}\|^2=\Sigma_{j=1}^{k_n}E(\xi_{nj\delta}-\eta_{nj\delta})^2E\|U_{nj\delta}\|^2$$

$$=2\Sigma_{j=1}^{k_n}P\{\|X_{nj}\|>\delta\}E\|X_{nj\delta}\|^2\leq \max_j E\|X_{nj\delta}\|^2\Sigma_{j=1}^{k_n}P\{\|X_{nj}\|>\delta\}.$$

Since for $\varepsilon<\delta$, by infinitesimality,

$$\lim\sup_{n\to\infty}\max_j E\|X_{nj\delta}\|\leq \lim\sup_{n\to\infty}\max_j E\|X_{nj\delta}-X_{nj\varepsilon}\|+E\|X_{nj\varepsilon}\|)$$

$$\lim_{n\to\infty}\delta\max_j P\{\|X_{nj}\|>\varepsilon\}+\varepsilon=\varepsilon,$$

we have that for every $\delta>0$

(3.8) $$\lim_{n\to\infty}\max_j E\|X_{nj\delta}\|=0.$$

Therefore, $E\|\Sigma_{j=1}^{k_n}(\xi_{nj\delta}-\eta_{nj\delta})U_{nj\delta}\|^2\to 0$

and in particular, by (3.6) the sequences

$$\{L(S_n)\}\quad\text{and}\quad\{L(\Sigma_{j=1}^{k_n}X_{nj\delta})*L(\Sigma_{j=1}^{k_n}(X_{nj}-X_{nj\delta}))\}$$

are weak-convergence equivalent. Thus, since $\{L(S_n)\}$ is relatively shift compact, the same is true for $\{L(\Sigma_{j=1}^{k_n}X_{nj\delta})\}$ for every $\delta>0$ (Parthasarathy (1967), III.2.2) i. e. $\{L(\Sigma_{j=1}^{k_n}\tilde{X}_{nj\delta})$ is tight. From this (3.5) follows at once.

The next is a unicity lemma for the Lévy-Khinchin representation in Hilbert space (it is true with much more generality). A nice proof of this result can be found in Parthasarathy (1967) pages 110-111.

3.10. Lemma. If

$$\delta_{a_1} * N(0,A_1) * c\mathrm{Pois}\mu_1 = \delta_{a_2} * N(0,A_2) * c\mathrm{Pois}\mu_2,$$

then $a_1=a_2$, $A_1=A_2$ and $\mu_1=\mu_2$ (provided $\mu_1\{0\}=\mu_2\{0\}$).

We end up this section with a lemma on tightness of sums of truncated variables. The result is stated for H-valued rv's but is true in more generality.

3.11. Lemma. Let $\{X_j\}_{j=1}^n$ be symmetric H-valued random variables and $S=\Sigma_{j=1}^n X_j$. Let $X_{j\delta}$ be the variables truncated at the level $\delta>0$ and $S_\delta=\Sigma_{j=1}^n X_{j\delta}$. Then for every compact convex symmetric set K and $\delta>0$,

$$P\{S_\delta \in K^c\} \leq 2P\{S \in K^c\}.$$

Proof. Let $U_{j\delta}, V_{j\delta}$ and $\xi_{j\delta}$ be a set of independent random variables with laws:

$$L(U_{j\delta}) = L(X_j \mid \|x\| \leq \delta)$$

$$L(V_{j\delta}) = L(X_j \mid \|x\| > \delta)$$

$L(\xi_{j\delta})$ = Bernoulli with expectation $P\{\|X_j\| \leq \delta\}$.
For simplicity, set

$$U=(U_{1\delta},\ldots,U_{n\delta}),\ V=(V_{1\delta},\ldots,V_{n\delta}),\ \xi=(\xi_{1\delta},\ldots,\xi_{n\delta})$$

and denote by . the usual inner product, e.g. $\xi.U=\Sigma_{j=1}^n \xi_{j\delta}U_{j\delta}$. Then, $L(X_{j\delta})=(\xi_{j\delta}U_{j\delta})$, $L(X_j)=L(\xi_{j\delta}U_{j\delta}+(1-\xi_{j\delta})V_{j\delta})$, $L(S_\delta)=L(\xi.U)$ and $L(S)=L(\xi.U+(I-\xi).V)$, where $I=(1,\ldots,1)$. So, if $T=\{0,1\}^n$, we have

$$P\{S\in K^c\}=P\{\xi.U+(I-\xi).V\in K^c\}$$

$$=\Sigma_{\tau\in T}P\{\xi.U+(I-\xi).V\in K^c|\xi=\tau\}P\{\xi=\tau\}$$

$$=\Sigma_{\tau\in T}P\{\tau.U+(I-\tau).V\in K^c\}P\{\xi=\tau\}$$

$$\geqslant \frac{1}{2}\Sigma_{\tau\in T}P\{\tau.U\in K^c\}P\{\xi=\tau\}$$

$$=\frac{1}{2}P\{\xi.U\in K^c\}=\frac{1}{2}P\{S_\delta\in K^c\},$$

where the last inequality is consequence of the following: if $\tau.U\in K^c$, either $\tau.U+(I-\tau).V\in K^c$ or $\tau.U-(I-\tau).V\in K^c$ (otherwise $2\tau.U\in 2K$) and both events have the same probability by symmetry.

4. The limit theorems

__4.1. Theorem.__ Let $\{X_j\}_{j=1}^n$ be independent H-valued rv's, $\mu_j=L(X_j)$, $S=\Sigma_{j=1}^n X_j$, $\mu=L(S)$, $X_{j\delta}=X_j\chi_{\{\|X_j\|\leqslant\delta\}}$, $a_\delta=\Sigma_{j=1}^n EX_{j\delta}$, and A_δ the covariance operator of $\Sigma_{j=1}^n X_{j\delta}$. Then

$$d_3[\mu,N(a_\delta,A_\delta)*\mathrm{Pois}(\Sigma_{j=1}^n\mu_j|\,\|x\|>\delta)]$$

$$\leqslant 2^{1+1/3}[\max_{1\leqslant j\leqslant n}\|EX_{j\delta}\|^2\Sigma_{j=1}^n P\{\|X_j\|>\delta\}]^{1/3}$$

$$+\ 3^{-1}(1+3^{3/4})\delta\,\mathrm{tr}A_\delta+2e^2\max_{1\leqslant j\leqslant n}P\{\|X_j\|>\delta\}\Sigma_{j=1}^n P\{\|X_j\|>\delta\}.$$

__Proof.__ Let $U_{j\delta}$, $V_{j\delta}$, $\xi_{j\delta}$ and $\eta_{j\delta}$ be independent random variables with laws:

$$L(U_{j\delta}) = L(X_j \mid \|x\| \leq \delta)$$

$$L(V_{j\delta}) = L(X_j \mid \|x\| > \delta)$$

$$L(\xi_{j\delta}) = L(\eta_{j\delta}) = \text{Bernoulli with expectation}$$

$$P\{\|X_j\| \leq \delta\},$$

as usual. Then

$$L(X_j) = L(\eta_{j\delta}U_{j\delta} + (1-\xi_{j\delta})V_{j\delta} + (\xi_{j\delta}-\eta_{j\delta})U_{j\delta}). \tag{4.1}$$

The equations (3.7) together with a simple computation give

$$\begin{aligned} &d_3[L(X),\ L(X+\Sigma_{j=1}^n(\xi_{j\delta}-\eta_{j\delta})U_{j\delta})]. \\ &\leq\ 2^{1+1/3}[\max_j E\|X_{j\delta}\|^2\Sigma_{j=1}^n P\{\|X_j\|>\delta\}]^{1/3}; \end{aligned} \tag{4.2}$$

also

$$\begin{aligned} &d_3(L(\Sigma_{j=1}^n\eta_{j\delta}U_{j\delta}),\ N(a_\delta,\ A_\delta)) \\ &= d_3(L(\Sigma_{j=1}^n X_{j\delta}),\ N(a_\delta, A_\delta)) \\ &= d_3(L(\Sigma_{j=1}^n X_{j\delta}-a_\delta\ ,\ N(0, A_\delta)) \\ &\leq 3^{-1}(1+3^{3/4})\delta\, \mathrm{tr} A_\delta \end{aligned} \tag{4.3}$$

by Theorem 3.2; and

$$\begin{aligned} &d_3(L(\Sigma_{j=1}^n(1-\xi_{j\delta})V_{j\delta}),\ \mathrm{Pois}(\Sigma_{j=1}^n\mu_j \mid \|x\|>\delta)) \\ &= d_3(L(\Sigma_{j=1}^n(X_j-X_{j\delta})),\ \mathrm{Pois}(\Sigma_{j=1}^n(\mu_j \mid \{\|x\|>\delta\})+P\{\|X_j\| \leq \delta\}\delta_0)) \\ &\leq 2e^2\Sigma_{j=1}^n P^2\{\|X_j\|>\delta\} \leq 2e^2\max_j P\{\|X_j\|>\delta\}\Sigma_{j=1}^n P\{\|X_j\|>\delta\} \end{aligned} \tag{4.4}$$

by Theorem 3.4 (it is easy to see that if μ and ν are p.m.'s then, if F is the set of bounded measurable functions, $\sup_{f\in F}|\int fd(\mu-\nu)| \leqslant 2\sup_{A\in B}|\mu(A)-\nu(A)|$). Now the theorem follows from (4.1)-(4.4) using Lemma 3.1.

<u>4.2. Theorem.</u> Let $\{X_{nj}\}$ be an infinitesimal system. In order that $\{L(S_n-x_n)\}$ converge weakly for some sequence $\{x_n\}\subset H$ it is necessary and sufficient that the following conditions be satisfied:

(i) $$\lim_{\delta\downarrow 0} \begin{matrix}\limsup \\ \liminf\end{matrix}_{n\to\infty} \Sigma_{j=1}^{k_n} E\langle X_{nj\delta}-EX_{nj\delta},y\rangle^2=\langle Ay,y\rangle$$

for some nuclear positive Hermitian operator A,

(ii) $$\lim_{N\to\infty}\limsup_{\delta\downarrow 0}\limsup_{n\to\infty}\Sigma_{j=1}^{k_n}\Sigma_{k=N+1}^{\infty}E\langle X_{nj\delta}-EX_{nj\delta},e_k\rangle^2=0$$

for some (all) cons $\{e_k\}$,

(iii) there exists a positive measure μ on H such that

$\int\min(1,\|x\|^2)d\mu(x)<\infty$ and

$$w\text{-}\lim_{n\to\infty}\Sigma_{j=1}^{k_n}L(X_{nj})|\{\|x\|>\delta\}=\mu|\{\|x\|>\delta\}$$

for every $\delta>0$ such that $\mu\{\|x\|=\delta\}=0$.

(iv) if $\alpha(x)=\min(1,\|x\|)\|x\|^{-1}x$, there exists $a\in H$ such that

$$\lim_{n\to\infty}\Sigma_{j=1}^{k_n}E\alpha(X_{nj}) - x_n=a.$$

Then the limit of $\{L(S_n-x_n)\}$ is $N(a,A)*c\mathrm{Pois}\mu$.

<u>Proof.</u> Assume (i)-(iv) are satisfied. Using a simple modification of the argument in the proof of Lemma 7.2.1 of Chung (1974) it is easy to show that if $a_{nm}^r\geqslant 0$ are such that $\lim_{m\to\infty}\limsup_{n\to\infty}a_{nm}^r=0$, r=1,...,h, then there exists a subsequence m_n such that $\lim_{n\to\infty}a_{nm_n}^r=0$ for every r=1,...,h. Hence, if $A_{n\delta}$

is the nuclear operator associated to the covariance of $\Sigma_{j=1}^{k_n}(X_{nj\delta}-EX_{nj\delta})$, the hypotheses imply the existence of a sequence $\delta_n \downarrow 0, \delta_n \leqslant 1$, such that

(a) $\lim_{n\to\infty}\delta_n \ \mathrm{tr}A_{n\delta_n}=0$,

(b) $\lim_{n\to\infty}d_1(\Sigma_{j=1}^{k_n}\mu_{nj}|\{\|x\|>\delta_n\},\mu|\{\|x\|>\delta_n\})=0$,

(c) $\lim_{n\to\infty}\int_{\|x\|>\delta_n}\alpha(x)d(\Sigma_{j=1}^{k_n}\mu_{nj}-\mu)(x)=0$,

(where $\mu_{nj}=L(X_{nj})$ as usual),

(d) $\lim_{n\to\infty}\max_{1\leqslant j\leqslant k_n}\| EX_{nj\delta_n}\|^2\Sigma_{j=1}^{k_n}P\{\|X_{nj}\|>\delta_n\}=0$,

(e) $\lim_{n\to\infty}\max_{1\leqslant j\leqslant k_n}P\{\|X_{nj}\|>\delta_n\}\Sigma_{j=1}^{k_n}P\{\|X_{nj}\|>\delta_n\}=0$,

(f) $\lim_{n\to\infty}\max_{1\leqslant j\leqslant k_n}P\{\|X_{nj}\|>\delta_n\}=0$.

(for (d), recall (3.8)).

Then, if $a_n=\Sigma_{j=1}^{k_n}E\alpha(X_{nj})$ and $a_{n\delta}=\Sigma_{j=1}^{k_n}E\alpha(X_{nj\delta})$,

$$d_3[L(S_n-a_n),\ N(0,A)*c\mathrm{Pois}\mu]$$

$$\leqslant d_3[L(S_n-a_n),\ N(0,A_{n\delta_n})*c\mathrm{Pois}(\Sigma_{j=1}^{k_n}\mu_{nj}|\|x\|>\delta_n)]$$

$$+\ d_3[N(0,A_{n\delta_n})*c\mathrm{Pois}(\Sigma_{j=1}\mu_{nj}|\|x\|>\delta_n),\ N(0,A)*c\mathrm{Pois}\mu].$$

The first summand at the right hand side tends to zero by Theorem 4.1 and the properties of $\{\delta_n\}$. For the second, we have

$$d_3[N(0,A_{n\delta_n})*c\mathrm{Pois}(\Sigma_{j=1}^{k_n}\mu_{nj}|\|x\|>\delta_n),\ N(0,A)*c\mathrm{Pois}\mu]$$

$$\leqslant d_3[N(0,A_{n\delta_n}),N(0,A)]+d_3[c\mathrm{Pois}(\Sigma_{j=1}^{k_n}\mu_{nj}|\|x\|>\delta_n),c\mathrm{Pois}\mu]$$

by a previous lemma. The second term at the right hand side tends to zero by Lemma 3.3, Lemma 3.6 and the properties of $\{\delta_n\}$. So, we need only prove that $d_3(N(0,A_{n\delta_n}), N(0,A))\to 0$.

Define A^N as $A^N(x)=\Sigma_{i=1}^N\langle x,e_i\rangle\pi_N A(e_i)$ where π_N is the orthogonal projection onto the subspace generated by $e_1,\ldots,e_N$, and likewise for $A_{n\delta}^N$. If $L(Z)=N(0,A)$ and $Z^N=\Sigma_{i=1}^N\langle Z,e_i\rangle e_i$, then $L(Z^N)=N(0,A^N)$. Let now $\|f'\|_\infty\leq 1$; then

$$|E(f(Z)-f(Z^N))| \leq E\|Z-Z^N\| \leq (E\|Z-Z^N\|^2)^{\frac{1}{2}}$$

$$= (\Sigma_{i=N+1}^\infty E\langle Z,e_i\rangle^2)^{\frac{1}{2}} = (\Sigma_{i=N+1}^\infty\langle Ae_1\ Ae_i,e_i\rangle)^{\frac{1}{2}}.$$

Therefore,

$$d_1(N(0,A^N),\ N(0,A)) \leq (\Sigma_{i=N+1}^\infty\langle Ae_i,e_i\rangle)^{\frac{1}{2}},$$

and analogously,

$$d_1(N(0,A_{n\delta}^N),\ N(0,A_{n\delta})) \leq (\Sigma_{i=N+1}^\infty\langle A_{n\delta}e_i,e_i\rangle)^{\frac{1}{2}}.$$

Hence, by the nuclearity of A and conditions (i) and (ii),

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} d_1(N(0,A_{n\delta}),N(0,A))$$

$$\leq \limsup_{N\to\infty}\limsup_{\delta\downarrow 0}\limsup_{n\to\infty} d_1(N(0,A_{n\delta}{}^N),N(0,A_{n\delta}))$$

$$+ \limsup_{N\to\infty}\limsup_{\delta\downarrow 0}\limsup_{n\to\infty} d_1(N(0,A_{n\delta}{}^N),N(0,A^N))$$

$$+ \limsup_{N\to\infty} d_1(N(0,A^N),\ N(0,A))=0.$$

For the second limit, note that hypothesis (i) implies that $\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\|A_{n\delta}{}^N-A^N\|=0$ for each N, and therefore $\lim_{\delta\downarrow 0}\limsup_{n\to\infty} d_1(N(0,A_{n\delta}{}^N),N(0,A^N))=0$ as these are Gaussian measures in dimension N.

So, there exists $\delta_n \downarrow 0$ which in addition to conditions (a)-(f) satisfies also

$$\lim_{n\to\infty} d_1(N(0,A_{n\delta_n}),N(0,A))=0.$$

This proves the direct part of the theorem.

We now proceed to the proof of the converse. By Theorem 3.9 the set of finite measures

$$d\nu_n(x) = \begin{cases} \Sigma_{j=1}^{k_n} \|x-EX_{nj1}\|^2 dL(X_{nj})(x) & \text{if } \|x\| \leqslant 1 \\ \Sigma_{j=1}^{k_n} dL(X_{nj})(x) & \text{if } \|x\| >1 \end{cases}$$

is relatively compact. Let $\{\nu_{n'}\}$ be a convergent subsequence and ρ its limit. Define μ as

$$d\,\mu(x)=\max(\|x\|^{-2},1)\,d\,\rho(x).$$

Then

$$\int \min(1,\|x\|^2)\,d\mu(x)<\infty,$$

i.e. μ is a Lévy measure.

The convergence of $\{\nu_{n'}\}$ to ρ easily implies that for every $\delta>0$ such that $\mu\{\|x\|=\delta\}=0$,

$$(4.5)\qquad \Sigma_{j=1}^{k_{n'}} L(X_{n'j}) \mid \{\|x\|>\delta\} \to_w \mu \mid \{\|x\|>\delta\} ,$$

$$(4.6)\qquad \lim_{\varepsilon\downarrow 0} \left\{ \begin{matrix} \limsup \\ \liminf \end{matrix} \right._{n'\to\infty} \Sigma_{j=1}^{k_n} E\|X_{n'j\varepsilon}-EX_{n'j1}\|^2=\rho\{0\}<\infty,$$

and from (4.6), using infinitesimality, one obtains that

$$(4.6)' \qquad \lim_{\varepsilon\downarrow 0} \left\{\begin{matrix}\limsup \\ \\ \liminf\end{matrix}\right\}_{n'\to\infty} \Sigma_{j=1}^{k_n} E\| X_{n'j\varepsilon} - EX_{n'j\varepsilon}\|^2 = \rho\{0\} < \infty .$$

(For the proof of (4.6)' use (3.8)).

Note that (4.5) proves (iii) along a subsequence (any subsequence n' such that $\{\nu_{n'}\}$ converges).

A simple computation shows that

$$\lim_{\varepsilon\downarrow 0} \limsup_{n'\to\infty} \left| \Sigma_{j=1}^{k_{n'}} [E<X_{n'j\varepsilon}-EX_{n'j\varepsilon}-EX_{n'j},y>2-\int_{|<x,y>|\leqslant\varepsilon} <x-\int_{|<x,y>}$$

$$x \, dL(X_{n'j})(x),y>^2 dL(X_{n'j})(x)$$

$$\leqslant \lim_{\varepsilon\downarrow 0} \limsup_{n'\to\infty} 3\varepsilon^2 \Sigma_{j=1}^{k_{n'}} P\{\| X_{n'j}\| \geqslant \varepsilon\} = 0$$

(to obtain the last limit just note that by (4.5),

$\varepsilon^2 \limsup_{n'\to\infty} \Sigma_{j=1}^{k_{n'}} P\{\|X_{n'j}\|>\varepsilon\} \leqslant \varepsilon^2 \mu\{\|x\|\geqslant \varepsilon\}$ and that since

$$\int \min(1,\|x\|^2)\, d\mu(x)<\infty, \varepsilon^2\mu\{\|x\|\geqslant\varepsilon\} \leqslant \int_{\varepsilon\leqslant \|x\| \leqslant \varepsilon^{\frac{1}{2}}} \|x\|^2 d\mu(x)$$

$$+\varepsilon\int_{\varepsilon^{\frac{1}{2}}<\|x\|\leqslant 1} \|x\|^2 d\mu(x) + \varepsilon^2\mu\{\|x\|>1\} \to 0 \quad \text{as } \varepsilon\downarrow 0).$$

Therefore (4.6)' can be applied in one dimension to obtain that for each $y \in H$ there exists a subsequence $\{n''\} \subset \{n'\}$ such that

$$\lim_{\varepsilon\downarrow 0} \left\{\begin{matrix}\limsup \\ \\ \liminf\end{matrix}\right\}_{n''\to\infty} \Sigma_{j=1}^{k_{n''}} E<X_{n''j\varepsilon}-EX_{n''j\varepsilon},y>^2 = \rho o y^{-1}\{0\} < \infty .$$

Since H is separable, by a diagonal argument we can obtain a subsequence $\{n''\} \subset \{n'\}$ such that the last limit holds simultaneously for a countable dense set $\{y_i\} \subset H$. Then an approximation argument shows that this holds for every $y \in H$ (note that (4.6)' implies that the bilinear forms

$\Sigma_{j=1}^{k_{n''}} E\langle X_{n''j\varepsilon} - EX_{n''j\varepsilon}, y\rangle\langle X_{n''j\varepsilon} - EX_{n''j\varepsilon}, z\rangle$ are uniformly bounded). So, (i) is proved along a subsequence $\{n''\}$ $\{n'\}$. Note that by (4.6)' the operator A defined as $\langle Ay,y\rangle = \rho o y^{-1}\{0\}$ is nuclear.

Next we prove (ii) along a subsequence. For each natural N define the pseudonorm $r_N^2(x) = \Sigma_{i=N+1}^{\infty}\langle x, e_i\rangle^2$. We note first that

$$\text{(4.7)}\qquad \lim_{\delta\downarrow 0}\limsup_{n'\to\infty}\Sigma_j |E[r_N^2(X_{n'j} - X'_{n'j})_\delta - r_N^2(X_{n'j\delta/2} - X'_{n'j\delta/2})]| = 0$$

where $L(X'_{nj}) = L(X_{nj})$ and the X'_{nj} are independent and independent of the X_{nj}. In fact,

$$|E[r_N^2(X_{nj} - X'_{nj})_\delta - r_N^2(X_{nj\delta/2} - X'_{nj\delta/2})]|$$

$$\leqslant E|r_N(X_{nj} - X'_{nj})_\delta + r_N(X_{nj\delta/2} - X'_{nj\delta/2})|$$

$$\times |r_N(X_{nj} - X'_{nj})_\delta - r_N(X_{nj\delta/2} - X'_{nj\delta/2})|$$

$$\leqslant 2\delta E r_N[(X_{nj} - X'_{nj})_\delta - X_{nj\delta/2} + X'_{nj\delta/2}] \leqslant 8\delta^2 P\{\|X_{nj}\| > \delta/2\},$$

But, as shown before in this same proof,

$$\lim_{\delta\downarrow 0}\limsup_{n'\to\infty}\Sigma_j \delta^2 P\{\|X_{n'j}\| > \delta\} = 0$$

and (4.7) is proved.

Hence, in order to prove (ii) along the subsequence $\{n'\}$, by (4.7) and the fact that $2Er_N^2(X_{nj\delta} - EX_{nj\delta}) = Er_N^2(X_{nj\delta} - X'_{nj\delta})$, it is enough to show that

$$(4.8)\qquad \lim_{N\to\infty}\limsup_{\delta\downarrow 0}\limsup_{n'\to\infty}\Sigma_{j=1}^{k_n} Er_N^2(\tilde{X}_{n'j\delta})=0$$

where $\tilde{X}_{nj\delta}$ is a symmetrization of $X_{nj\delta}$. But the Lévy and converse Kolmogorov inequalities give that for every $\tau>0$,

$$(4.9)\qquad \Sigma_{j=1}^{k_n} r_n^2(\tilde{X}_{n'j\delta}) \leqslant [(\tau+\delta)^2+\tau^2/2]/[1-4P\{r_N(\tilde{S}_{n',\delta})>\tau\}]$$

where $\tilde{S}_{n,\delta}=\Sigma_{j=1}^{k_n}\tilde{X}_{nj\delta}$. Since $\{L(\tilde{S}_n)\}$ is relatively compact Lemma 3.11 implies that the family of p.m.'s $\{L(\tilde{S}_{n,\delta})$: $n=1,\ldots;\delta>0\}$ is tight, hence that

$$\lim_{N\to\infty}\sup_{n,\delta}P\{r_N(\tilde{S}_{n,\delta})>\tau\}=0$$

for every $\tau>0$. So, taking limits in (4.9) we obtain (4.8), i.e. condition (ii) for the subsequence $\{n'\}$.

So, (i)-(iii) hold along $\{n''\}$ and this means that if μ is as defined and $<Ay,y>=\rho o y^{-1}\{0\}$, then,

$$\text{w-}\lim_{n''\to\infty}L(S_{n''}-\Sigma_{j=1}^{k_n}E\alpha(X_{n''j}))=N(0,A)*c\text{Pois}\mu,$$ and that moreover, $x_{n''}-\Sigma_{j=1}^{k_n}E\alpha(X_{n''j})=a$ and $L(S_{n''}-x_{n''})\to_w N(a,A)*c\text{Pois}\mu$.

Now, by the unicity Lemma 3.10, if for any other subsequence $\{m'\}$ the sequence $\{\nu_{m'}\}$ converges, then the limit is also ρ, which means that $\{\nu_n\}$ is convergent and that the conditions (i)-(iv) hold by the previous arguments.

Next we will give a corollary on Gaussian convergence (and leave it for the interested reader to obtain results on Poisson convergence). Although a direct proof of the following result based on the lemmas in section 3 and using truncation is also possible, we prefer to rely on the previous theorem.

4.3. Corollary. Let $\{X_{nj}\}$ be an infinitesimal array of H-valued rv's and $S_n=\Sigma_{j=1}^{k_n}X_{nj}$. Then there exists $\{x_n\}\subset H$ such that $\{L(S_n-x_n)\}$ converges weakly to a Gaussian p.m. if and only if:

(i) for some (every) $\delta>0$ and every $y\in H$

$$\lim_{n\to\infty}\Sigma_{j=1}^{k_n}E<X_{nj\delta}-EX_{nj\delta},y>^2=<Ay,y>$$

where A is a nuclear positive Hermitian operator,

(ii) for every $\varepsilon>0$, $\lim_{n\to\infty}\Sigma_{j=1}^{k_n}P\{\|X_{nj}\|>\varepsilon\}=0$

(iii) for some (every) $\delta>0$ and some (every) cons $\{e_i\}$

$$\lim_{N\to\infty}\lim\sup_{n\to\infty}\Sigma_{j=1}^{k_n}\Sigma_{k=N+1}^{\infty}E<X_{nj\delta}-EX_{nj\delta},e_i>^2 = 0.$$

In this case, there exists $a\in H$ such that

$$\lim_{n\to\infty}\Sigma_{j=1}^{k_n}EX_{nj} -x_n=a \quad \text{and}$$

$$w\text{-}\lim_{n\to\infty}L(S_n-x_n)=N(a,A).$$

Proof. Assume the first three conditions hold. Note that condition (ii) will allow for the centering $\Sigma_{j=1}^{k_n}EX_{nj}$ instead of $\Sigma_{j=1}^{k_n}E\alpha(X_{nj1})$. So, by Theorem 4.2, it will be enough to show that condition (ii) in 4.2 holds. Let us let $\sim$ denote symmetrization as before, and r_N be as in the previous proof. In view of condition (ii), 4.3, the proof of (4.7) shows that

$$\lim\sup_{n\to\infty}\Sigma_j\left|E[\,r_N^2((\tilde{X}_{nj})_\delta)-2r_N^2(X_{nj\delta/2}-EX_{nj\delta/2})]\right| = 0 \tag{4.10}$$

for every $\delta>0$. So we have that

$$\lim_{N\to\infty}\lim\sup_{n\to\infty}\Sigma_jEr_N^2((\tilde{X}_{nj})_\delta) = 0$$

and since $r_N^2((\widetilde{X}_{nj})_\delta)$ decreases as $\delta\downarrow 0$,

$$\lim_{N\to\infty}\limsup_{\delta\downarrow 0}\ \limsup_{n\to\infty}\Sigma_j Er_N^2((\widetilde{X}_{nj})_\delta) = 0$$

Now another application of (4.10) gives condition (ii) in (4.2). The direct part is proved.

Conversely, if $\{L(S_n-x_n)\}$ converges to a Gaussian p.m. Theorem 4.2 already implies condition (ii). By condition (ii), 4.3,

$$\limsup_{n\to\infty}\left|\Sigma_{j=1}^{k_n}(E<X_{nj\delta}-EX_{nj\delta},y>^2-E<X_{nj\delta'}-EX_{nj\delta'},y>^2)\right|=0$$

as a simple computation shows. Therefore, condition (i) 4.2 is equivalent to (i) 4.3 in this case. So we need only prove condition (iii), 4.3.

By Theorem 4.2,

$$\lim_{N\to\infty}\limsup_{\delta\downarrow 0}\limsup_{n\to\infty}\Sigma_{j=1}^{k_n}Er_N^2(X_{nj\delta}-EX_{nj\delta}) = 0,$$

and by (4.10),

$$\lim_{N\to\infty}\limsup_{\delta\downarrow 0}\ \limsup_{n\to\infty}\Sigma_{j=1}^{k_n}Er_N^2((\widetilde{X}_{nj})_\delta) = 0.$$

Now, if $\delta'<\delta$,

$$\Sigma_{j=1}^{k_n}Er_N^2((\widetilde{X}_{nj})_\delta)\leq\Sigma_{j=1}^{k_n}Er_N^2((\widetilde{X}_{nj})_{\delta'})+\Sigma_{j=1}^{k_n}\int_{\delta'\leq\|\widetilde{X}_{nj}\|\leq\delta}r_N^2((\widetilde{X}_{nj}))dP,$$

but the second sum at the right hand side is bounded above by $\delta^2\Sigma_{j=1}^{k_n}P\{\|\widetilde{X}_{nj}\|>\delta'\}\to 0$ as $n\to\infty$ (note that $L(S_{nj})$ converges also to a Gaussian p.m.). Therefore

$$\limsup_{n\to\infty}\Sigma_{j=1}^{k_n}Er_N^2((\widetilde{X}_{nj})_\delta)\leq\limsup_{n\to\infty}\Sigma_{j=1}^{k_n}Er_N^2((\widetilde{X}_{nj})_{\delta'}),$$

and since $r_N^2((\widetilde{X}_{nj})_\delta)\geq r_N^2((\widetilde{X}_{nj})_{\delta'})$, we conclude that

$\lim_{N\to\infty} \limsup_{n\to\infty} \Sigma_{j=1}^{k_n} Er_N^2((\tilde{X}_{nj})_\delta)=0$ for every $\delta>0$. So, condition (iii), 4.3 follows from (4.10).

The last result is essentially due to Varadhan (1962) (Parthasarathy (1967) Theorem VI.6.3). The only difference is that one of his conditions (condition (4) in Parthasarathy VI. 6.3) does not appear in our version.

Among other consequences to Theorem 4.2 that can be easily derived we have the Lévy-Khinchin representation of infinitely divisible laws in Hilbert space and results on convergence of infinitely divisible laws (Varadhan (1962)). We will not pursue this subject.

Finally we just indicate briefly how to prove a result on approximation of $L(S_n)$ by $c\text{Pois}(\Sigma_j L(X_{nj}))$ (up to centerings).

The same Lindeberg type proof of Theorem 3.2 yields:

<u>4.5. Theorem.</u> Let $\{X_i\}_{i=1}^n$ be as in Theorem 3.2. Then

$$d_3[L(S),\text{Pois}(\Sigma_{i=1}^n L(X_1))]\leq KC\,\text{tr}A$$

for some universal constant $K>0$.

This theorem together with Theorem 3.4 leads to an estimate of the type of Theorem 4.1 for $d_3[L(S-a_\delta)$, $\text{Pois}\ \Sigma_j L(X_j-a_{j\delta})]$ where $a_\delta=\Sigma_j a_{j\delta}$ and $a_{j\delta}=EX_{j\delta}$. Then, the type of arguments already exposed in this paper give the following theorem, which is one of the main results in Varadhan (1962) (or Parthasarathy (1967), chapter VI).

<u>4.5. Theorem.</u> Let $\{X_{nj}\}$ be an infinitesimal array, $S_n=\Sigma_{j=1}^{k_n} X_{nj}$ and for $\delta>0$, $a_{nj\delta}=EX_{nj\delta}$ and $a_{n\delta}=\Sigma_{j=1}^{k_n} EX_{nj\delta}$. Then $\{L(S_n)\}$ is relatively shift compact if an only if

$\{(\mathrm{Pois}\Sigma_{j=1}^{k_n} L(X_{nj}-a_{nj\delta}))*\delta_{a_{n\delta}}\}$ is, and in this case

$$\lim_{n\to\infty} d_3[L(S_n-a_{n\delta}),\ \mathrm{Pois}\Sigma_{j=1}^{k_n} L(X_{nj}-a_{nj\delta})] = 0.$$

References

[1] DE ACOSTA, A., ARAUJO, A. and GINÉ, E. (1977). "On Poisson measures, Gaussian measures and the central limit theorem in Banach spaces". (To appear in Adv. in Probability, Vol IV).

[2] DE ACOSTA, A. and SAMUR, J. (1977). "Infinitely divisible laws and the converse Kolmogorov inequality in Banach space". I.V.I.C. Departamento de Matemáticas, Fascículo 2. (To appear in Studia Math).

[3] ARAUJO, A. and GINÉ, E. (1976). "Type, cotype and Lévy measures on Banach spaces".(To appear in Ann. Probability).

[4] BILLINGSLEY, P. (1968). "Convergence of probability measures". J. Wiley and Sons, New York.

[5] CHUNG, K.L. (1974). "A course in Probability". Assoc. Press. New York.

[6] DUDLEY, R.M. (1966). "Convergence of Baire measures". Studia Math. 27 251-268.

[7] FERNIQUE, X. (1970). "Integrabilité des vecteurs gaussiens". C.R. Acad. Sci. Paris, Ser. A 270 1698-1699.

[8] FREEDMAN (1974). "The poisson approximation for dependent events". Ann. Probability 2 256-269.

[9] KAHANE, J.P. (1968). "Some random series of functions". Heath. Lexington, Mass.

[10] KUELBS, J. and KURTZ, T. "Berry Esseen estimates in Hilbert space and an application to the LIL." Ann. Probability 2, 387-407.

[11] LeCAM, L. (1970). "Remarques sur le thérème limite central dans les spaces localement convexes". Les probabilités sur les structures algébriques, 233-249. C.N.R.S., Paris.

[12] LANG, S. (1962). "Introduction to differentiable manifolds". John Wiley & Sons, New York.

[13] PARTHASARATHY, P.K. (1967). "Probability measures in metric spaces". Acad. Press, New York.

[14] RAO, R. RANGA. (1962). "Relations between weak and uniform convergence of measures with applications". Ann. Math. Statistics 33. 659-680.

[15] VARADHAN, S.R.S. (1962). "Limit theorems for sums of independent random variables with values in a Hilbert space". Sankhya 24 213-238.

(1) Instituto Venezolano de Investigaciones Científicas and Universitat Autónoma de Barcelona.

(2) Universidad Central de Venezuela.