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AN EXISTENCE AND STABILITY THEOREM FOR A CLASS OF FUNCTIONAL EQUATIONS

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ABSTRACT

Consider the class of functional equations

$$g[F(x,y)]=H[g(x),g(y)],$$

where $g:E\to X$, $F:E\times E\to E$, $H:X\times X\to X$, E is a set and (X,d) is a complete metric space. In this paper we prove that, under suitable hypotheses on F, H and $\delta(x,y)$, the existence of a solution of the functional inequality

$$d(f[F(x,y)],H[f(x),f(y)])\leqslant\delta(x,y),$$

implies the existence of a solution of the above equation.

1.- D.H. Hyers in [2] proved the following stability theorem for the Cauchy equation:

If $f:E\to E'$, E, E' Banach spaces, and there exists $\delta>0$ such that

$$\|f(x+y)-f(x)-f(y)\|\leqslant\delta \qquad \text{for every } x,y\in E,$$

then there exists a unique additive function $g:E\to E'$ such that $\|g(x)-f(x)\|\leqslant\delta$ for every $x\in E$.

The existence of an additive function is part of the theorem above; therefore it can be also view as an existence theorem.

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In the present paper we consider the following class of functional equations

$$(\#) \qquad g[F(x,y)] = H[g(x),g(y)],$$

where $F : E\times E \to E$, $H: X\times X \to X$ are given functions, $g: E \to X$ is the unknown function, E is a set and (X,d) is a complete metric space.

We prove, under certain hypotheses on F and H, that the existence of a solution of the functional inequality.

$$d(f[F(x,y)], H[f(x),f(y)]) \leqslant \delta(x,y),$$

where $\delta: E\times E \to R^+$ is a suitable function, implies the existence of a solution of the equation (#).

If $\delta(x,y)=\text{const}$, we obtain as a corollary, a generalization of the Hyers theorem of stability.

2.- Consider the functional equation (#). We put $G(x) =F(x,x)$, $K(u) = H(u,u)$ and we assume that X be a modulus set for K, i. e. $K(X) = X$ (see [3]), and K is invertible.

For a non-negative integer n, G^n and K^n will denote the n-th iterates of G and K respectively; K^{-n} denotes the n-th iterate of K^{-1}.

For every $x\in E$, we define $x^o = x$, $x^n = G^n(x)$.

For every $f:E\to X$ and every $x,y\in E$, we set $\delta(x,y) = d(f[F(x,y)], H[f(x),f(y)])$; instead of $\delta(x^n,y^n)$ we write $\delta_n(x,y)$ and instead of $\delta_n(x,x)$ we write $\delta_{n+1}(x)$.

We prove the following existence theorem:

THEOREM 1.- Assume that:

(i) there exists a function $k:R^+\to R^+$ strictly increasing, superadditive, such that for every $u,v\in X$, $d(K(u),K(v))=k(d(u,v))$, moreover for some constant $c>1$, $k(t)\geqslant ct$ for every $t\in R^+$;

(ii) for every $u,v\in X$, $H[H(u,v),H(u,v)]=H[H(u,u),H(v,v)]$;

(iii) H is continuous;

(iv) for every $x,y\in E$, $F[F(x,y),F(x,y)]=F[F(x,x),F(y,y)]$.

If there is a function $f:E\to X$ such that for every $x,y\in E$, $\delta_n(x,y)=o(c^n)$ and the series $\sum_{n=1}^{\infty} c^{-n}\delta_n(x)$ converges, and for some $\tilde{x},\tilde{y}\in E$ it is $\liminf_{n\to+\infty} k^{-n}[d(f(\tilde{x}^n),f(\tilde{y}^n))]>0$, then the equation (#) has a non constant solution.

PROOF.- Let $k_1(t)=k(t)$ and $k_m(t_1,\dots,t_m)=k[t_m+k_{m-1}(t_1,\dots,t_{m-1})]$. Fix $x\in E$; then for every $n\geqslant 1$ it is

$$(1)\qquad d(f(x^n),K^n[f(x)])\leqslant \delta_n(x)+k_{n-1}[\delta_1(x),\dots,\delta_{n-1}(x)].$$

This follows by induction: for n=2 we have

$$d(f(x^2),K^2[f(x)])\leqslant d(f(x^2),K[f(x^1)])+d(K[f(x^1)],K^2[f(x)])=\delta_2(x)+k[\delta_1(x)],$$

by (i); assume that (1) holds for n-1, then

$$d(f(x^n),K^n[f(x)])\leqslant d(f(x^n),K[f(x^{n-1})])+d(K[f(x^{n-1})],K^n[f(x)]) =$$

$$=\delta_n(x)+k[d(f(x^{n-1}),K^{n-1}[f(x)])]\leqslant \delta_n(x)+k\{\delta_{n-1}(x)+k_{n-1}[\delta_1(x),..,\delta_{n-1}(x)]\}=$$

$=\delta_n(x)+k_{n-1}[\delta_1(x),\dots,\delta_{n-1}(x)]$, since k is increasing.

Now we define the sequence of functions $\{q_n\}$ by $q_n(x)=K^{-n}[f(x^n)]$; q_n is defined for every $n\geqslant 0$, since X is a modulus set for K.

We claim that, for each $x\in E$, $\{q_n(x)\}$ is a Cauchy-sequence in X.

Let n>m, then

$$d(q_n(x),q_m(x))=d(K^{-n}[f(x^n)],K^{-m}[f(x^m)])=k^{-n}[d(f[G^{n-m}(x^m)],K^{n-m}[f(x^m)])]$$

(k^{-n} is the n-th iterate of k^{-1}, the inverse function of k); k^{-n} is subadditive and increasing and $\delta_j(x^m)=\delta_{j+m}(x)$, therefore

$$d(q_n(x),q_m(x))\leqslant k^{-n}\{\delta_n(x)+k_{n-m-1}[\delta_{m+1}(x),\dots,\delta_{n-1}(x)]\}\leqslant$$

$$\leqslant k^{-n}[\delta_n(x)]+k^{-n+1}[\delta_{n-1}(x)]+\dots+k^{-m-1}[\delta_{m+1}(x)]\leqslant \sum_{j=m+1}^{n} c^{-j}\delta_j(x),$$

the convergence of the series $\sum_{n=1}^{\infty} c^{-n}\delta_n(x)$ implies that $\{q_n(x)\}$ is a Cauchy-sequence.

X is complete, thus $\{q_n\}$ converges pointwise to a function g.

We prove that g is a non constant solution of the equation (#). For every $x,y\in E$, we have

$$d(K^{-n}\{f[F(x^n,y^n)]\},K^{-n}\{H[f(x^n),f(y^n)]\}) =$$

$$=d(K^{-n}\{f[G^n\{F(x,y)\}]\},H\{K^{-n}[f(x^n)],K^{-n}[f(y^n)]\})\to d(g[F(x,y)],H[g(x),g(y)])$$

as $n\to+\infty$, by the hypotheses (ii), (iii) and (iv); moreover because $\delta_n(x,y)=o(c^n)$,

$$d(K^{-n}\{f[F(x^n,y^n)]\},K^{-n}\{H[f(x^n),f(y^n)]\})=$$

$$=k^{-n}[d(f[F(x^n,y^n)],H[f(x^n),f(y^n)])]=k^{-n}[\delta_n(x,y)]\leqslant c^{-n}\delta_n(x,y)\to 0 \quad \text{as } n\to+\infty,$$

thus $g[F(x,y)]=H[g(x),g(y)]$.

g is not constant, since $d(g(\tilde{x}),g(\tilde{y}))=\lim_{n\to+\infty} d(K^{-n}[f(\tilde{x}^n)],K^{-n}[f(\tilde{y}^n)])=$ $=\lim_{n\to+\infty} k^{-n}[d(f(\tilde{x}^n),f(\tilde{y}^n))]>0$ by the hypothesis.

REMARK 2.- It is possible to substitute the hypotheses on the sequences $\{\delta_n(x,y)\}$ and $\{\delta_n(x)\}$ by the following ones:

a) for every $x\in E$, the series $\sum_{j=1}^{\infty} k^{-j}[\delta_j(x)]$ converges;

b) for every $x,y\in E$, $\lim_{n\to+\infty} k^{-n}[\delta_n(x,y)]=0$.

The hypotheses a) and b) allow us to drop the assumption $k(t)\geqslant ct$ for some $c>1$.

COROLLARY 3.- In the hypotheses of Theorem 1, there exists a unique solution g of the functional equation (#) such that for every $x\in E$, $d(g(x),f(x))\leqslant \sum_{n=1}^{\infty} k^{-n}[\delta_n(x)]$.

Furthermore if E is a topological space, G is continuous and $\sup\{\delta_n(x,y):\ n\geqslant 0,\ x,y\in E\}=\delta<+\infty$, then the continuity of f implies that of g.

PROOF.- If $\{q_n\}$ is defined as in theorem 1,

$$d(q_n(x),f(x))=d(K^{-n}[f(x^n)],f(x))=k^{-n}[d(f(x^n),K^n[f(x)])]\leqslant$$

$$\leqslant k^{-n}\{\delta_n(x)+k_{n-1}[\delta_1(x),\ldots,\delta_{n-1}(x)]\}\leqslant \sum_{j=1}^{\infty} k^{-j}[\delta_j(x)].$$

Let $n\to+\infty$, we have $d(g(x),f(x))\leqslant \sum_{j=1}^{\infty} k^{-j}[\delta_j(x)]$.

Let $\tilde{g}:E\to X$ be a solution of the equation (#), such that for every $x\in E$, $d(\tilde{g}(x),f(x))\leqslant \sum_{j=1}^{\infty} k^{-j}[\delta_j(x)]$ and $g(z)\neq\tilde{g}(z)$ for some $z\in E$.

We have $\tilde{g}(z^n)=K^n[\tilde{g}(z)]$, $g(z^n)=K^n[g(z)]$, hence

$$d(\tilde{g}(z^n),g(z^n))=d(K^n[\tilde{g}(z)],K^n[g(z)])=k^n[d(\tilde{g}(z),g(z))],$$

and

$$d(\tilde{g}(z^n),g(z^n))\leqslant d(\tilde{g}(z^n),f(z^n))+d(g(z^n),f(z^n))\leqslant 2\sum_{j=1}^{\infty} k^{-j}[\delta_j(z^n)]\leqslant$$

$$\leqslant 2k^n\{\sum_{j=n+1}^{\infty} k^{-j}[\delta_j(z)]\};$$

thus $k^n[d(\tilde{g}(z),g(z))]\to+\infty$ as $n\to+\infty$ and

$k^n[d(\tilde{g}(z),g(z))]\leqslant 2k^n\{\sum_{j=n+1}^{\infty} k^{-j}[\delta_j(z)]\}$, so $d(\tilde{g}(z),g(z))\leqslant$

$\leqslant 2\sum_{j=n+1}^{\infty} k^{-j}[\delta_j(z)]\to 0$ as $n\to+\infty$, a contradiction.

If $\sup\{\delta_n(x,y):\ n\geqslant 0,\ x,y\in E\}=\delta<+\infty$, we have $d(q_n(x),q_m(x))$ $\leqslant \sum_{j=m+1}^{n} c^{-j}\delta$, $n>m$, so $\{q_n\}$ converges uniformely to g. If E is a topological space, since K^{-1} is continuous at 0 and $d(K^{-1}(u),K^{-1}(v))=$ $=K^{-1}[d(u,v)]$, K^{-1} is continuous, then the continuity of G and f implies that q_n is continuous for every n. Thus, because of the uniform convergence, g is continuous.

EXAMPLES.- 1) Consider the functional equation

$$g[T_a(x)+T_a(y)+\psi]=T_a[g(x)]+T_a[g(y)],$$

where $g:L^1(R)\to L^1(R)$, $x,y,\psi\in L^1(R)$, $a\in R$, T_a is the translation operator related to a, that is $(T_a(x))(t)=x(t-a)$.

Obviously the hypotheses (i)-(iv) of Theorem 1 are satisfied.

Let $Q:L^1(R)\to L^1(R)$ be a linear operator which commute with T_a. Then

$$\|Q[T_a(x)+T_a(y)+\psi]-T_a(Qx)-T_a(Qy)\|_1=\|Q\psi\|_1.$$

We can now apply Corollary 3 and obtain the solution

$$g(x)=Qx+\sum_{k=1}^{\infty}2^{-k}T_{-ka}[Q\psi].$$

2) Consider the Cauchy equation

$$g(x+y)=g(x)+g(y),$$

where $g:Z\to Z$. Let $f:Z\to Z$ be defined by $f(x)=[x/2]$ ([t] is the integral part of t); then $|f(x+y)-f(x)-f(y)|\leqslant 1$, but for every additive function g, the difference g-f is unbounded. Corollary 3 fails because Z is not a modulus set for $K(u)=2u$ (see [4]).

3) Consider the functional equation

$$g(2\sqrt[3]{xy^2}+1)=2\sqrt[3]{g(x)[g(y)]^2},$$

where $g:R\to R$. In this case the hypothesis (iv) of Theorem 1 is not satisfied. Let $f(x)=x$, we can construct the sequence

$$q_n(x)=x+1-2^{-n},$$

$q_n(x)\to x+1$ as $n\to+\infty$, but the function $g(x)=x+1$ is not a solution of the equation.

4) Consider the functional equation

$$g(2\sqrt[3]{xy^2})=2\sqrt[3]{g(x)[g(y)]^2}+1,$$

where $g:R\to R$. In this case the hypothesis (ii) of Theorem 1 is not satisfied. Let $f(x)=x$, we can construct the sequence

$$q_n(x)=x-1+2^{-n},$$

$q_n(x)\to x-1$ as $n\to+\infty$, but the function $g(x)=x-1$ is not a solution of the equation.

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