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# ON SOME FUNCTIONAL EQUATIONS FROM ADDITIVE AND NONADDITIVE MEASURES - III

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In this series, this paper is devoted to the study of two related functional equations primarily connected with weighted entropy and weighted entropy of degree β (which are weighted additive and weighted β additive respectively) which include as special cases Shannon's entropy, inaccuracy (additive measures) and the entropy of degree β (nonadditive) respectively. These functional equations which arise mainly from the representation and these 'additive' properties are solved for fixed m and n (positive integers) by simple and direct methods.

## Introduction

Let $(\Omega,A,P)$ be a probability space. Consider an experiment E, that is, a finite measurable partition (of events) $\{E_1,\ldots,E_n\}$ $(n>1)$ of $\Omega$ with the objective probabilities of these events given by $P(E_i)=p_i \geqslant 0$ for every $E_i$ such that $P = (p_i)\in\Delta_n$ where $\Delta_n = \{P = (p_i,\ldots,p_n): p_i \geqslant 0, \sum_1^n p_i = 1\}$. The different events $E_i$ depend upon the experimenter's goal or upon some qualitative characteristic of the physical system taken into consideration;

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that is, they have different weights (or utilities). In order to distinguish the events $E_1,\dots,E_n$ with respect to a given qualitiative characteristic of the physical system taken into account, ascribe to each event $E_i$ a non-negative number $W(E_i) = w_i$ ($\geqslant 0$) directly proportional to its importance and call $w_i$ the weight (utility) of the event $E_i$.

Then the weighted entropy of the experiment E is defined as [3]:

(1) $$H_n(P;W) = H_n(p_1,\dots,p_n;w_1,\dots,w_n) = -\sum_{i=1}^{n} w_i p_i \log p_i,$$

and the weighted entropy of degree $\beta$ ($\neq 1$) is defined as [2]:

(2) $$H_n(P;W) = (2^{1-\beta}-1)^{-1}\sum_{i=1}^{n} w_i(p_i^{\beta}-p_i)$$

where $P\in\Delta_n$, $W = (w_1,\dots,w_n)$ with $w_i \geqslant 0$.

It is easy to see that i) $H_n^{\beta} \to H_n$ as $\beta\to 1$; ii) If $w_i = w$ for all i, then (1) gives the Shannon entropy $H_n(P) = c\Sigma p_i \log p_i$ and (2) gives the entropy of degree $\beta$ [4], $H_n^{\beta}(P) = c(\Sigma p_i^{\beta}-1)$; iii) If $w_i = \frac{q_i}{p_i}$ (under suitable conditions) with $\Sigma q_i = 1$, (1) becomes the inaccuracy [9], $I_n(P\|Q) = -\Sigma p_i \log q_i$.

There are so many algebraic properties which are satisfied by them [1,2,3,4,9]. In particular, the weighted entropies can be represented in the form of sums as

$$H_n(P;W) = \sum_i f(p_i,w_i),$$

and

$$H_n^{\beta}(P;W) = \sum_i g(p_i,w_i)$$

and possess the following properties:

weighted additivity

$$H_{mn}(P*Q;W*V) = \sum_{j=1}^{m} q_j v_j \cdot H_n(P;W) + \sum_{i=1}^{n} p_i w_i \cdot H_m(Q;V)$$

weighted $\beta$-additivity

$$H_{mn}^{\beta}(P*Q;W*V) = \sum_j q_j v_j \cdot H_n^{\beta}(P;W) + \sum_i p_i^{\beta} w_i \cdot H_m^{\beta}(Q;V)$$

respectively, where $Q \in \Delta_m$, $P*Q = (p_i q_j)$, $V = (v_1, \dots, v_m)$, $W*V = (w_i v_j)$, $\beta(\neq 1)$ is real.

The above two properties lead to the study of the functional equations

$$(3) \quad \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j, w_i v_j) = \sum_j q_j v_j \cdot \sum_i f(p_i, w_i) + \sum_i p_i w_i \cdot \sum_j f(q_j, w_j)$$

and

$$(4) \quad \sum_{i=1}^{n} \sum_{j=1}^{m} g(p_i q_j, w_i v_j) = \sum q_j v_j \cdot \sum_i g(p_i, w_i) + \sum_i p_i^{\beta} w_i \cdot \sum_j g(q_j, w_j),$$

$\beta \neq 1$. So, a characterization of (1) or (2) can be obtained, by determining all the solutions of (3) or (4). In the next section, we solve the functional equation.

$$(5) \quad \sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j, u_i v_j) = \sum_j q_j v_j \cdot \sum_i h(p_i, u_i) + \sum_i p_i^{\alpha} u_i \cdot \sum_j h(q_j, v_j),$$

for all $\alpha \in R$ (reals), which obviously includes both (3) and (4), where $P = (p_i) \in \Delta_n$, $Q = (q_j) \in \Delta_m$, $U = (u_1, \dots, u_n)$, $V = (v_1, \dots, v_m)$ with $u_i, v_j \geqslant 0$. As a matter of fact, we determine all the solutions of the functional equation (5) holding for some (arbitrary but) fixed pair (m,n) when the function h is Lebesque measurable using simple methods adopted in [6,7].

Weighted Entropies: Solutions of the equation (5)

Let $I = [0,1]$, $I_1 = [0,1]$, $R$, reals, $R^+$, non-negative reals. We follow the convention $0 \log 0 = 0$, $0^\alpha = 0$, $1^\alpha = 1$. In order to solve (5), we make use of the following result in [8]:

Result 1. Let $G_{ij}: I \times I \to R$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m)$ be measurable in each variable and satisfy the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} G_{ij}(p_i, q_j) = 0 \tag{6}$$

$(P = (p_i) \in \Delta_n,\ Q = (q_j) \in \Delta_m)$ holding for some fixed pair $m,n$ $(\geqslant 3)$. Then $G_{ij}$ are given by

$$\begin{aligned} G_{ij}(p,q) &= G_{ij}(p,0) - \sum_{\ell=1}^{m} G_{i\ell}(p,0)q + G_{ij}(0,q) - \sum_{k=1}^{n} G_{kj}(0,q)p \\ &+ \sum_{k=1}^{n} G_{kj}(0,0)p + \sum_{\ell=1}^{m} G_{i\ell}(0,0)q - \sum_{k}\sum_{\ell} G_{k\ell}(0,0)pq - G_{ij}(0,0). \end{aligned} \tag{7}$$

Let $h: I \times R^+ \to R$ be measurable in each variable and satisfy the functional equation (5), for a fixed pair $m,n$ $(\geqslant 3)$.

For arbitrary, but fixed $U = (u_i)$, $V = (v_j)$, by defining

$$G_{ij}(p,q) = h(pq, u_i v_j) - v_j q h(p, u_i) - p^\alpha u_i h(q, v_j) \tag{8}$$

for $p,q \in I$, $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,m)$, (5) can be reduced to (6) with $G_{ij}$ measurable in each variable, so that Result 1, applies and (7) holds.

With $p_1 = 1 = q_1$, $p_i = 0 = q_j$ $(i,j \geqslant 2)$, $u_i, v_j$ arbitrary (5) becomes

$$\begin{aligned} h(1,u_1 v_1) &+ \sum_{\ell=2}^{m} h(0,u,v_\ell) + \sum_{k=2}^{n} \sum_{j=1}^{m} h(0,u_k v_j) \\ &= v_1[h(1,u_1) + \sum_{2}^{n} h(0,u_k)] + u_1[h(1,v_1) + \sum_{2}^{m} h(0,v_\ell)]. \end{aligned} \tag{9}$$

By letting $u_1 = 1 = v_1$ in (9), we get,

$$\sum_{k=2}^{n} \sum_{\ell=2}^{m} h(0,u_k v_\ell) = h(1,1).$$

Since $u_k, v_\ell$ are arbitrary, it follows that

$$(m-1)(n-1)h(0,u) = h(1,1) = c \quad \text{(say)} \tag{10}$$

for all $u \in R^+$.

Setting $u_1 = 1$ in (9) and using (9) and (10), we obtain

$$c(1-v_1) = 0, \text{ that is, } c = 0$$

since $v_1$ is an arbitrary non-negative real number. Thus

$$h(0,u) = 0, \quad \text{for all} \quad u \in R^+. \tag{11}$$

Now, from (8) and (11), we have

$$G_{ij}(p,0) = 0 = G_{ij}(0,q), \quad \text{for } p,q \in I,$$

that is, from (7), it follows that $G_{ij}(p,q) = 0$ for all $p,q \in I$. Thus, from (7) and (8) results,

$$h(pq,uv) = vqh(p,u)+p^{\alpha}uh(q,v), \tag{12}$$

for all $p,q \in I$, $u,v \in R^+$.

By interchanging p and q and u and v respectively in (12), we get

$$h(p,u)(q^{\alpha}-q)v = h(q,v)(p^{\alpha}-p)u. \tag{13}$$

Thus, when $\alpha \neq 1$, from (13) results

(14) $$h(p,u) = cu(p^{\alpha}-p), \quad u \in R^{+},\ p \in I,$$

where c is an arbitrary constant.

Now, let us consider the case $\alpha = 1$. With $u = 1 = u$, (12) becomes

$$h(pq,1) = qh(p,1)+ph(q,1),$$

that is, by the measurability of h and by (11), we have

(15) $$h(p,1) = ap \log p, \quad \text{for } p \in I,$$

where a is an arbitrary constant.

From (15) and (12) with $v = 1$, we obtain

$$h(pq,u) = qh(p,u)+apqu \log q$$

$$\text{also} = ph(q,u)+apqu \log p,$$

that is,

$$\frac{h(p,u)-apu \log p}{p} = \frac{h(q,u)-aqu \log q}{q}$$

independent of p and q and depends only on u

$$= g(u) \quad (\text{say}), \quad \text{for } p,q \in I_1,$$

and $u \in R^{+}$. Hence,

(16) $$h(p,u) = apu \log p + pg(u), \quad p,q \in I_1, \quad u \in R^{+},$$

where g is measurable.

Use (12) and (16) to get

$$(17) \qquad g(uv) = ug(v)+vg(u), \qquad u,v \geqslant 0.$$

Since g is measurable, from (17) results

$$(18) \qquad g(u) = bu \log u, \qquad u > 0,$$

where b is an arbitrary constant.

Setting $u_i = 0 = v_j$ in (5), we have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j, 0) = 0,$$

that is, by [5],

$$(19) \qquad h(p,q) = a_1 p + a_2, \qquad \text{for } p \in I$$

with $a_1 + mna_2 = 0$.

Since $h(0,0) = 0$ by (1), we see that $a_2 = 0 = a_1$, so that (18) holds for $u = 0$ also. Hence, from (16) and (19), we have

$$(20) \quad h(p,u) = apu \log p + bpu \log u, \quad \text{for} \quad p \in I,\ u \in R^+,$$

where a and b are arbitrary constants.

Thus, we have proved the following theorem:

<u>Theorem.</u> Let $h: I \times R^+ \to R$ be measurable in each variable and satisfy the functional equation (5) for some fixed pair $m,n \geqslant 3$. Then h is given either by (14) when $\alpha \neq 1$ or by (20) when $\alpha = 1$.

<u>Remark</u>. Unfortunately, because of the occurrence of the weights in the right side of the equation (5), the solutions are not dependent on m and n.

Further, for $u = 1$ or constant, (14) reduces to a result proved in [5], connected with the entropy of degree $\beta$.

## References

[1] N.L. AGGARWAL and C.F. PICARD, "Functional equations and information measures with preference", Kybernetika, 14, 1978, 174-181.

[2] H. EMPTOZ, "Information de type $\beta$ intégrant un concept d'utilité",C.R. Acad. Sci. Paris 282 A, 1976, 911-914.

[3] S. GIASU, "Weighted entropy", Reports on Mathematical Physics 2, 1971, 165-179.

[4] J. HAVRDA and F. CHARVAT, "Quantification method of classification processes", Kybernetika 3, 1967, 30-35.

[5] PL. KANNAPPAN, "On a generalization of sum from functional equation - I" (to appear).

[6] PL. KANNAPPAN, "On some functional equations from additive and nonadditive measures - I" (to appear).

[7] PL. KANNAPPAN, "On some functional equations from additive and nonadditive measures - II" (to appear).

[8] PL. KANNAPPAN, "On a generalization of sum from functional equation - III (submitted).

[9] D.F. KERRIDGE, "Inaccuracy and inference", J. Roy. Statist. Soc. Ser. B 23, 1961, 184-194.

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