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RANDOM LINES AND TESSELLATIONS IN A PLANE

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ABSTRACT

Our purpose is the study of the so called "mixed random mosaics", formed by superposition of a given tessellation, not random, of congruent convex polygons and a homogeneous Poisson line process. We give the mean area, the mean perimeter and the mean number of sides of the polygons into which such mosaics divide the plane.

1. Introduction

Lines in the euclidean plane E_2 are parametrized by (p,θ), the polar coordinates of the foot of the perpendicular from the origin to the line. The density element for lines, invariant under euclidean motions, is $dp\wedge d\theta$. The measure of the set of lines intersecting a convex set K is equal to the length L of the boundary ∂K. For a line segment of length b this measure is equal to 2b, since the segment must be considered as a flattened convex set.

The standard homogeneous Poisson line process of intensity λ is that line process corresponding to a homogeneous Poisson point process of constant intensity λ in the strip $\{(p,\theta);0\leqslant p<\infty,\ 0\leqslant\theta<2\pi\}$. The fundamental property of this line process is that the number m of lines of the process hitting a convex set K has a Poisson distribution of intensity λL. Moreover, each line of the plane intersects the lines of the process in a linear homogeneous point process of Poisson of intensity 2λ. This property also holds for the intersection of a line of the process with the other lines of the process. For all these questions see, for instance Kendall-Moran [9], Solomon [18], or [17].

A Poisson system of random lines of intensity λ partitiones the plane into a random tessellation (or mosaic) of simple convex polygons with almost surely each vertex being a vertex of four polygons of the tessellation. These random polygons were first studied by Goudsmidt [6] who obtained the mean number of sides, the mean perimeter, the mean area and the mean area squared of the polygons. More general results were obtained later by Miles [10], [12] and Richards [15]. Interesting and fundamental questions referring to ergodicity and edge effects have been carefully treated by Cowan [3], [4] and Ambartzumian [1], [2] (see also Miles [11], [13]).

Our purpose is the study of mixed random tesselations of the plane, originated when a Poisson system of random lines is superposed on a preexistent tessellation, not random, formed by compact, convex, congruent polygons which cover the whole plane without overlapping. For instance, figs.1, 2 represent the case of tessellations of congruent pentagons crossed by a random Poisson system of lines.

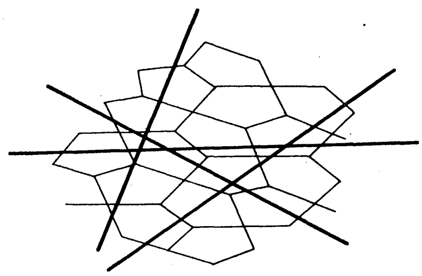

FIG.1

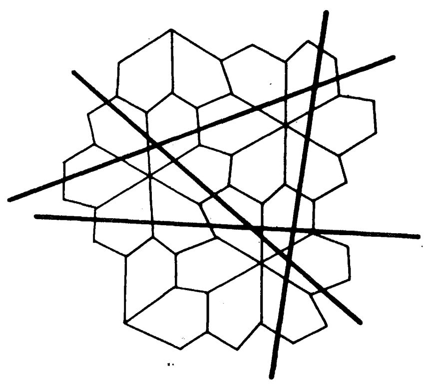

FIG.2

We obtain the mean are (5.3), the mean number of sides (5.4) and the mean perimeter (5.5) of the resulting polygons. It seems to be an interesting open question to find the second order moments of these characteristics for that kind of mixed mosaics.

2. Tessellations of compact, congruent, convex polygons.

We use the term tessellation for any arrangement of bounded, convex, congruent polygons (called the fundamental polygons or cells of the tessellation) fitting together so as to cover the whole plane without overlapping. Examples of tessellations and their relations to group theory can be seen in the books of Coxeter [5], Guggenheimer [8] or in Grünbaum-Shephard [7].

Assume a given tessellation T whose fundamental polygons have area f, perimeter u and number of sides n (equal to the number of vertices). It is known that the only possible values of n are 3,4,5,6. Consider a circle $Q(R)$ of radius R (which we will assume sufficiently large) and let $\nu(\partial Q)$ be the number of fundamental polygons which are intersected by the boundary ∂Q and $\nu(Q)$ the number of fundamental polygons within Q. Then we have

$$\lim_{R\to\infty} \frac{\nu(\partial Q)}{\nu(Q)} = 0. \tag{2.1}$$

For a proof, notice that if D denotes the diameter of a fundamental polygon, we have (for large R),

$$\nu(\partial Q) \leqslant \frac{\pi(R+D)^2 - \pi(R-D)^2}{f} = \frac{4\pi RD}{f}, \quad \nu(Q)\ f \geqslant \pi(R-D)^2$$

and (2.1) follows.

This relation (2.1) or the weaker one $\nu(\partial Q)/\pi R^2 \to 0$ (as $R\to\infty$) makes possible to eliminate the "edge effects" in some passage to the limit which we shall perform later. It is worthy to note that in the hyperbolic plane these "edge effects" are not negligible, so that the passage from a finite region to the whole plane in some tessellation problems, must be treated with care (see [16]).

Let now n_k denote the number of vertices of each fundamental polygon which are surrounded by k faces of the tessellation ($k \geqslant 3$). We shall need the following identities

$$\sum n_k = n \ , \qquad \sum n_k \left(\frac{1}{2} - \frac{1}{k}\right) = 1 \tag{2.2}$$

where the sums are extended over all values of k. The first equality is nothing else than the definition of n_k and the second is an easy consequence of the Euler relation vertices-sides+faces=1, applied to the bounded planar graph formed by the edges of the tessellation within Q for $R \to \infty$.

3. Some results of stochastic geometry

Consider a circle $Q=Q(R)$ of radius R. The random variable $m=m(Q)=$ number of lines of a given homogeneous Poisson line process of intensity λ hitting Q has a Poisson $2\pi R\lambda$-distribution (see Miles [10] or Santaló [17]; notice that in [17] we use $\lambda/2$ instead of the present λ).

Therefore we have the following moments

$$\begin{aligned} E(m) &= 2\pi R\lambda \ , \quad E(m^2) = 2\pi R\lambda + (2\pi R\lambda)^2 \\ E(m^3) &= 2\pi R\lambda + 3(2\pi R\lambda)^2 + (2\pi R\lambda)^3 \\ E(m^4) &= 2\pi R\lambda + 7(2\pi R\lambda)^2 + 6(2\pi R\lambda)^3 + (2\pi R\lambda)^4 \end{aligned} \tag{3.1}$$

Moreover, we know that for m random lines (in the sense of geometrical probability) which meet a circle Q, the mean number of intersection points n_P which are inside Q is [17, p. 53]

$$E(n_P | m) = m(m-1)/4 \tag{3.2}$$

and thus, applying (3.1)

$$E(n_P) = EE(n_P|\, m) = \pi^2R^2\lambda^2. \tag{3.3}$$

We also know that [17, p. 54]

$$E(n_P^2|m) = \frac{1}{2}\binom{m}{2} + \frac{3}{2}\binom{m}{4} + \frac{16}{\pi 2}\binom{m}{3} \tag{3.4}$$

and thus

$$E(n_P^2) = \pi^2R^2\lambda^2 + \pi^4R^4\lambda^4 + \frac{64}{3}\pi R^3\lambda^3 . \tag{3.5}$$

From (3.3) and (3.5) we have, as $R\to\infty$,

$$\lim E(\frac{n_P}{\pi R^2}) = \pi\lambda^2 \quad , \quad \lim E(\frac{n_P}{\pi R^2})^2 = \pi^2\lambda^4 \tag{3.6}$$

so that, for any realization of the process, we have, almost sure,

$$\lim_{R\to\infty} \frac{n_P}{\pi R^2} = \pi\lambda^2. \tag{3.7}$$

Where n_P is the number of intersections of the lines of the Poisson process within Q.

4. Tessellations and Random Lines.

Consider a fixed tessellation T with the characteristics specified in n.2, and a Poisson line process P of intensity λ superposed to it. The tessellation T and the line process P define on the plane a mixed random mosaic M. Consider a circle Q(R) of large radius R. Let n_P denote the number of intersection points of the lines of the process within Q, n_T the number of vertices of the tessellation within Q and n_{PT} the number of intersections of lines of the process with sides of the tessellation within Q. From a classical Crofton's formula of integral geometry ([17], p. 31) we know that $E(n_{PT}|m) = mL/\pi R$, where L is the total length of the sides of the tessellation within Q and m denotes the number of lines of the process intersecting Q. Neglecting edge effects, for R sufficiently large, we can take $\pi R^2/f$ as the number of fundamental polygons intersected by Q and thus we have

$$(4.1)\qquad E(n_{PT}|\,m) = \frac{uRm}{2f} \quad , \quad E(n_{PT}) = \frac{\pi R^2 u\lambda}{f} \quad , \quad E(\frac{n_{PT}}{\pi R^2}) = \frac{u\lambda}{f} \ .$$

For n_T and R large, with the notations of n . 2, we have

$$(4.2)\qquad \frac{n_T}{\pi R^2} = \frac{1}{f} \sum_k \frac{n_k}{k} \lambda^2 .$$

Thus, using (3.3), we have

$$(4.3)\qquad E\,(\frac{n_P n_T}{(\pi R^2)^2}) = \frac{\pi}{f} \sum_k \frac{n_k}{k} \lambda^2 \ .$$

The second moments involving n_P and n_{PT}, other than $E(n_P^2)$, are not easy to calculate exactly. However, we shall give some upper bounds.

Noting that n_{PT}, for each line of the process P and large R is less than the number of fundamental poligons within a rectangle of sides 2R and 2D (where D means the diameter of the fundamental polygons), we have $n_{PT} < (4RD/f)m$ and thus, using (3.1)

$$(4.4)\qquad E(\frac{n_{PT}}{\pi R^2})^2 < \frac{32D^2}{\pi f^2} (\frac{\lambda}{R} + 2\pi\lambda^2)$$

On the other hand we have $n_P \leqslant m(m-1)/2$ so that $E(n_{PT}n_P/(\pi R^2)^2) < 2Dm^2(m-1)/\pi^2 R^3 f$ and using (3.1)

$$(4.5)\qquad E\,(\frac{n_{PT}n_P}{(\pi R^2)^2}) < \frac{16D}{f} (\frac{\lambda^2}{R} + \pi\lambda^3) \ .$$

The total number of vertices v of the mixed random mosaic M generated by the union of T and P which are inside Q, is $v = n_P + n_{PT} + n_T$. Therefore, using (3.3), (3.5), (4.1), (4.2), (4.3), (4.4), and

(4.5) we have

$$(4.6)\qquad E\left(\frac{v}{\pi R^2}\right) = \pi\lambda^2 + \frac{u\lambda}{f} + \frac{1}{f}\sum_k \frac{n_k}{k}$$

$$(4.7)\qquad E\left(\frac{v^2}{(\pi R^2)^2}\right) < \frac{\lambda^2}{R^2} + \pi^2\lambda^4 + \frac{64\lambda^3}{3\pi R} + \frac{32D^2}{\pi f^2}\left(\frac{\lambda}{R} + 2\pi\lambda^2\right)$$

$$+ \frac{1}{f^2}\sum_k \left(\frac{n_k}{k}\right)^2 + \frac{32D}{f}\left(\frac{\lambda^2}{R} + \pi\lambda^3\right)$$

$$+ \frac{2\pi\lambda^2}{f}\sum_k \frac{n_k}{k} + \frac{2u\lambda}{f^2}\sum_k \frac{n_k}{k}$$

and, as $R\to\infty$, we have

$$\lim_{R\to\infty} \operatorname{var}\,(v^2/(\pi R^2)^2) < \left(\frac{32\pi D}{f} - \frac{2\pi u}{f}\right)\lambda^3 + \left(\frac{64D^2}{f^2} - \frac{u^2}{f^2}\right)\lambda^2 .$$

5. Mean values of the area, perimeter and number of sides of the cells of a mixed random mosaic.

Consider the mixed random mosaic M of n.4. composed of the tessellation T of congruent polygons and a homogeneous Poisson line process P of intensity λ. With the notations of the foregoing paragraph we have that tha number of edges e of M within the circle Q of radius R is

$$(5.1)\qquad e = 2n_P + 2n_{PT} + \frac{1}{2}\,\frac{\pi R^2 n}{f}$$

where $n = \sum n_k$ is the number of sides of the fundamental polygons of the tessellation T. This equality disregards some "edge effects" on the boundary of Q which may be neglected for R sufficiently large. Therefore, as $R\to\infty$, we have

$$\lim_{R\to\infty} E(e/\pi R^2) = 2\pi\lambda^2 + 2u\lambda/f + n/2f. \tag{5.2}$$

By Euler's relation, we have

$$n_P + n_{PT} + (\pi R^2/f) \sum_k (n_k/k) - e + c = 1$$

where c denotes the number of cells of M within Q. Taking expectation and using (2.2) we deduce

$$\lim_{R\to\infty} E(c/\pi R^2) = \pi\lambda^2 + (u/f)\lambda + 1/f.$$

Thus, as $R\to\infty$, we have almost surely $\lim(\pi R^2/c) = (\pi\lambda^2+u\lambda/f + 1/f)^{-1}$ and the mean area of the cells of the mixed mosaic results

$$E(A) = \frac{f}{\pi f\lambda^2 + u\lambda + 1}. \tag{5.3}$$

For the mean number of sides of each region, we have

$$E(N) = \lim_{R\to\infty} (2e/c) = \frac{4\pi f\lambda^2 + 4u\lambda + n}{\pi f\lambda^2 + u\lambda + 1}. \tag{5.4}$$

Note that for n=4 (tessellation of quadrilaterals) is E(N)=4, independently of λ.

In order to find the mean perimeter S we note that the total length of all sides of T within Q is $(\pi R^2/2f)u$ and the mean value of the total length of the chords that the lines of the process P intercept in Q is $\pi^2R^2\lambda$ ([17], p. 30). Therefore the mean value of the total length of the sides of the mosaic within Q is $\pi R^2(u/2f+\pi\lambda)$ and the mean perimeter (as $R\to\infty$) is given by

$$E(S) = \frac{2\pi f\lambda + u}{\pi f\lambda^2 + u\lambda + 1}. \tag{5.5}$$

Note that for $\lambda=0$ the mixed mosaic reduces to the tessellation

T (not random) and for $f \to \infty$ it reduces to the well known random division of the plane by a Poisson line process [10].

6. Example

Considere the tessellation of parallelograms of the fig.3.

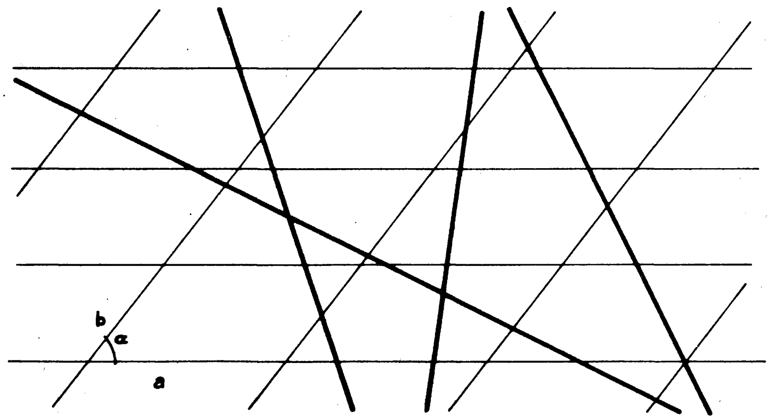

FIGURA 3

We have

$$f = a\, b \sin \alpha \quad , \quad u = 2(a+b), \quad n = n_4 = 4$$

and therefore

$$E(A) = \frac{ab \sin \alpha}{\pi ab \sin \alpha \lambda^2 + 2(a+b)\lambda + 1} \quad , \quad E(N) = 4$$

$$E(S) = \frac{2\pi ab \sin \alpha\, \lambda + 2(a+b)}{\pi ab \sin \alpha \lambda^2 + 2(a+b)\lambda + 1} .$$

If $a \to \infty$ we have the plane divided by parallel lines at distance $\Delta = b \sin \alpha$. Thus, a homogeneous Poisson line process of intensity λ determines on the plane on which are ruled parallel lines at a distance Δ apart, a mixed random mosaic of characteristics

$$E(A) = \frac{\Delta}{\pi\Delta\lambda^2+2\lambda} , \quad E(N)= 4 , \quad E(S) = \frac{2\pi\Delta\lambda + 2}{\pi\Delta\lambda^2 + 2\lambda} .$$

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