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ON THE STABILITY OF MULTIPOLAR ELASTIC MATERIALS

by

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1. Introduction.

In 1964, Green and Rivlin [1,2] proposed two non-standard theories of continua. Both papers concerned non-simple materials the first considered deformation gradients of higher order than the first as dependent variables; and the second, which generalised the first, treated materials whose kinematic state was not completely determined by the deformation function, but was also dependent upon some multipolar deformation functions. In both theories the existence of higher order stresses is fundamental.

In this paper, we present some prliminary results for stability and instability in these theories in the special but important case of elastic materials.

In section 2, we describe the two theories of Green and Rivlin and we obtain in a form more suitable for our purpose the equations of motions for the respective elastic materials. In section 3, we consider boundary conditions for the material for which there exists a conservation of energy.

In section 4 and 5, we discuss stability and instability respectively, and show that conservation of energy is an important factor. Thus by employing the energy as a Liapounov function, we obtain results concerning stability of the multipolar continuum in a similar manner to classical non-polar elasticity (c.f. Knops and Wilkes [5]). Instability is then investigaty by means of the concavity technique developed by Knops, Levine and Payne [3], and applied by them to non-linear classical elasticity.

2. Thermodynamics and the equations of motion.

We consider a body B whose particles X are identified with their positions X_A with respect to rectangular Cartesian co-ordinates in some reference configuration κ. We suppose these particles are deformed to positions x_i at time t.

Following Green and Rivlin [2], we further suppose that the kinematic state of the body is not fully determined by the monopolar deformation but is also dependent upon multipolar deformations $x_{iA_1\ldots A_\alpha}$ $(1 \leqslant \alpha \leqslant \nu)$ which are functions of material position and time.

We also adopt the expression for the kinetic energy per unit mass used by Green and Rivlin; thus

$$K(t) = \frac{1}{2}\dot{x}_i\dot{x}_i + \frac{1}{2}\sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu}\dot{x}_{iA_1\ldots A_\alpha}Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\dot{x}_{iB_1\ldots B_\beta} \tag{2.1}$$

where $Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}$ is a tensor function of material position satisfying

$$Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta} = Y_{B_1\ldots B_\beta:A_1\ldots A_\alpha} \tag{2.2}$$

At this point we make no assumptions concerning the definiteness of the kinetic energy but later in our discussion of stability we shall assume it to be non-negative.

The two major postulates of Green and Rivlin's theory are the balance of energy equation and the entropy production inequality, the first and second laws of thermodynamics. In the abscence of heat supply, body force and multipolar body forces, the laws can be expressed in the following forms,

$$\frac{d}{dt}\int_P \{\rho_o U + \frac{1}{2}\rho_o \dot{x}_i \dot{x}_i + \frac{1}{2}\rho_o \sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} \dot{x}_{iA_1\ldots A_\alpha} Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta} \dot{x}_{iB_1\ldots B_\beta}\} dV$$

$$= \int_{\partial P} \{p_i \dot{x}_i + \sum_{\alpha=1}^{\nu} p_{iA_1\ldots A_\alpha} \dot{x}_{iA_1\ldots A_\alpha} - h\} \, dA \qquad (2.3)$$

and

$$\frac{d}{dt}\int_P \rho_o S \, dV \geq - \int_{\partial P} \frac{h}{T} \, dA, \qquad (2.4)$$

where P is an arbitrary material volume of the body B in the reference consiguration κ and ∂P is its boundary. In these equations ρ_o is the density of the reference configuration, U is the internal energy, p_i is the monopolar stress vector per unit area of the reference configuration, $p_{iA_1\ldots A_\alpha}$ $(1 \leq \alpha \leq \nu)$ are multipolar stress vectors per unit of the reference configuration, and h is the heat flux per unit area of the reference configuration, S is the specific entropy and T the absolute temperature.

In this paper we consider only multipolar continua which are also elastic. We thus assume as our constitutive postulate that the free energy ψ, defined by

$$\psi = U - TS, \qquad (2.5)$$

the entropy, the heat flux, the monopolar stress and the multipolar stresses are dependent upon the temperature T, the deformation gradient $x_{i,A}$, the multipolar deformations $x_{iB_1\ldots B_\beta}$ and their gradients $x_{iB_1\ldots B_\beta,A}$ for $1 \leq \beta \leq \nu$. Additionally we assume that the heat flux can depend upon the temperature gradient $T_{,A}$, and that the stress vectors and the heat flux can depend upon the unit outward normal N_A to the surface ∂P. Thus

$$\psi = \psi(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T) \qquad (2.6)$$

$$S = S(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T) \qquad (2.7)$$

$$h = h(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T, T_{,A}, N_A) \tag{2.8}$$

$$p_k = p_k(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T, N_A) \tag{2.9}$$

$$p_{kA_1\ldots A_\alpha} = p_{kA_1\ldots A_\alpha}(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T, N_A) \text{ for } 1 \leqslant \alpha \leqslant \nu. \tag{2.10}$$

For these constitutive restrictions it can be shown in an analogous way to Green and Rivlin [2] that there exist monopolar and multipolar stress tensors, π_{Bk} and $\pi_{BkC_1\ldots C_\gamma}$ respectively, and a heat flux vector Q_B which satisfy

$$\pi_{Bk} = \pi_{Bk}(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T), \tag{2.11}$$

$$\pi_{BkC_1\ldots C_\gamma} = \pi_{BkC_1\ldots C_\gamma}(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T) \text{ for } 1 \leqslant \gamma \leqslant \nu, \tag{2.12}$$

$$Q_B = Q_B(x_{i,A}, x_{iB_1\ldots B_\beta}, x_{iB_1\ldots B_\beta,A}, T, T_{,A}) \tag{2.13}$$

and that the following relations hold:

$$h = Q_A N_A, \tag{2.14}$$

$$p_i = \pi_{Ai} N_A, \tag{2.15}$$

$$p_{iA_1\ldots A_\alpha} = \pi_{BiA_1\ldots A_\alpha} N_B \quad \text{for } 1 \leqslant \alpha \leqslant \nu, \tag{2.16}$$

$$S = -\frac{\partial\psi}{\partial T}, \tag{2.17}$$

$$\pi_{Ai} = \rho_o \frac{\partial\psi}{\partial x_{i,A}}, \tag{2.18}$$

$$\pi_{AiB_1\ldots B_\beta} = \rho_o \frac{\partial\psi}{x_{iB_1\ldots B_\beta,A}} \quad \text{for } 1 \leqslant \beta \leqslant \nu \tag{2.19}$$

$$\pi_{Ai,A} = \rho_o \ddot{x}_i, \tag{2.20}$$

$$\pi_{AiB_1\ldots B_\beta,A} = \rho_o \frac{\partial\psi}{\partial x_{iB_1\ldots B_\beta}} + \rho_o \sum_{\alpha=1}^{\nu} Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta} \ddot{x}_{iA_1\ldots A_\alpha}$$

$$\text{for } 1 \leqslant \beta \leqslant \nu \tag{2.21}$$

and

$$Q_A T_{,A} \leqslant 0. \tag{2.22}$$

In parallel with the general theory of multipolar elastic continua outlined above, we shall also consider a higher order gradient theory of elasticity in which the multipolar deformations are defined by

$$x_{iA_1 \ldots A_\alpha} = x_{i,A_1 \ldots A_\alpha} \quad \text{for } 1 \leqslant \alpha \leqslant \nu. \tag{2.23}$$

Our constitutive assumptions are now of the form

$$\psi = \psi(x_{i,A},\ x_{i,A_1 \ldots A_\alpha},\ T) \text{ etc.}, \tag{2.24}$$

and for these materials we find, by working analogously to Green and Rivlin [1], that in place of equations (2.18) - (2.21), we have

$$\pi_{Ai} + \pi_{BiA,B} = \rho_o \frac{\partial\psi}{\partial x_{i,A}} + \rho_o \sum_{\beta=1}^{\nu} Y_{A:B_1 \ldots B_\beta}\, \ddot{x}_{i,B_1 \ldots B_\beta}, \tag{2.25}$$

$$\pi_{(AiB_1 \ldots B_\beta)} + \pi_{CiAB_1 \ldots B_\beta,C} = \rho_o \frac{\partial\psi}{\partial x_{i,AB_1 \ldots B_\beta}} + \rho_o \sum_{\gamma=1}^{\nu} Y_{AB_1 \ldots B_\beta : C_1 \ldots C_\gamma}\, \ddot{x}_{i,C_1 \ldots C_\gamma} \quad \text{for } 1 \leqslant \beta \leqslant \nu - 2, \tag{2.26}$$

$$\pi_{(AiB_1 \ldots B_\beta)} = \rho_o \frac{\partial\psi}{\partial x_{i,AB_1 \ldots B_\beta}} + \rho_o \sum_{\gamma=1}^{\nu} Y_{AB_1 \ldots B_\beta : C_1 \ldots C_\gamma}\, \ddot{x}_{i,C_1 \ldots C_\gamma} \quad \text{for } \beta = \nu - 1, \tag{2.27}$$

where $\pi_{(AiB_1 \ldots B_\beta)}$ is the completely symmetric part of the stress tensor $\pi_{AiB_1 \ldots B_\beta}$ which by definition is already symmetric with respect to $B_1, \ldots, B_\beta$. Otherwise the theory remains unchanged.

3. Boundary conditions and conservation of energy.

For simplicity, from here on, we shall consider only isothermal processes of the body, so that we shall be studying a purely mechanical theory of multipolar elastic continua.

It follows from equations (2.18) - (2.21) that the time derivative of the total mechanical energy E(t) of the motion, defined by

$$E(t) = \int_{B} \rho_o \left[\frac{1}{2} \dot{x}_i \dot{x}_i + \frac{1}{2} \sum_{\alpha=1}^{\nu} \sum_{\beta=1}^{\nu} Y_{A_1 \ldots A_\alpha : B_1 \ldots B_\beta} \dot{x}_{iA_1 \ldots A_\alpha} \dot{x}_{iB_1 \ldots B_\beta} + \psi \right] dV \tag{3.1}$$

is given by

$$\frac{d}{dt} E(t) = \int_{\partial B} p_i \dot{x}_i dA + \int_{\partial B} \sum_{\beta=1} p_{iB_1 \ldots B_\beta} \dot{x}_{iB_1 \ldots B_\beta} dA \tag{3.2}$$

where the free energy and all the stress tensors are subject to constitutive equations of the form

$$\psi = \psi(x_{i,A}, x_{iB_1 \ldots B_\beta}, x_{iB_1 \ldots B_\beta, A}). \tag{3.3}$$

As the right hand side of (3.2) is comprised only of an integral over the boundary of the body, by choosing appropriate boundary conditions for the motion of the body we can make

$$\frac{d}{dt} E(t) \equiv 0, \tag{3.4}$$

so that the energy E(t) of the body is constant (=E(0)) for all time.

The boundary conditions that give rise to a constant energy include those of the form

$$\dot{x}_i = 0 \text{ on } \partial B_1 \quad , \; p_i = 0 \quad \text{on} \quad \partial B_2 \, , \tag{3.5}$$

$$\dot{x}_{iA_1 \ldots A_\alpha} = 0 \text{ on } \partial B_1^{(\alpha)}, \; p_{iA_1 \ldots A_\alpha} = 0 \text{ on } \partial B_2^{(\alpha)}, \; 1 \leqslant \alpha \leqslant \nu, \tag{3.6}$$

where ∂B_1 and $\partial B_1^{(\alpha)}$, $1 \leqslant \alpha \leqslant \nu$, are of positive measure if non-empty, and

$$\partial B_1 \cup \partial B_2 = \partial B\ ,\ \partial B_1 \cap \partial B_2 = \phi\ , \tag{3.7}$$

$$\partial B_1^{(\alpha)} \cup \partial B_2^{(\alpha)} = \partial B\ ,\partial B_1^{(\alpha)} \cap \partial B_2^{(\alpha)} = \phi\ ,\ 1 \leqslant \alpha \leqslant \nu. \tag{3.8}$$

Further, if $\partial B_1 = \phi$, we shall exclude rigid body translations by assuming

$$\int_B (x_i - X_i)\,dV = 0. \tag{3.9}$$

For the higher order gradient theory of elasticity, it can be shown in a similar manner to the above using (2.25) - (2.27) that the total mechanical energy E(t), defined by

$$E(t) = \int_B \rho_0 [\tfrac{1}{2}\dot{x}_i \dot{x}_i + \tfrac{1}{2}\sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta}\dot{x}_{i,A_1\ldots A_\alpha}\dot{x}_{i,B_1\ldots B_\beta} + \psi]\,d$$

$$\equiv E(0) \tag{3.10}$$

for boundary conditions of the form

$$\dot{x}_i = 0 \text{ on } \partial B_1,\ p_i = 0 \text{ on } \partial B_2 \tag{3.11}$$

$$\dot{x}_{i,A_1\ldots A_\alpha} = 0 \text{ on } \partial B_1^{(\alpha)},\ p_{iA_1\ldots A_\alpha} = 0 \text{ on } \partial B_2^{(\alpha)}, 1 \leqslant \alpha \leqslant \nu, \tag{3.12}$$

where ∂B_1, ∂B_2, $\partial B_1^{(\alpha)}$, $\partial B_2^{(\alpha)}$ satisfy the same conditions as above.

4. Stability.

In this section, we shall discuss the Liapounov stability of the null solution with respect to the reference configuration for the multipolar elastic continuum, the equations of motion for which we have discussed in the two previous sections.

We shall denote monopolar and multipolar displacements by u_i and $u_{iA_1 \dots A_\alpha}$ respectively, where

$$u_i = x_i - X_i \tag{4.1}$$

$$u_{iA_1 \dots A_\alpha} = x_{iA_1 \dots A_\alpha} - X_{iA_1 \dots A_\alpha} \tag{4.2}$$

and $X_{iA_1 \dots A_\alpha}$ is the value of $x_{iA_1 \dots A_\alpha}$ in the reference configuration.

The energy functional can now be written as

$$E(t) \equiv \int_B \rho o[\frac{1}{2}\dot{u}_i \dot{u}_i + \frac{1}{2} \sum_{\alpha=1}^{\nu} \sum_{\beta=1}^{\nu} Y_{A_1 \dots A_\alpha : B_1 \dots B_\beta} \dot{u}_{iA_1 \dots A_\alpha} \dot{u}_{iB_1 \dots B_\beta} + \psi]dV \tag{4.3}$$

where the free energy ψ is determined by the constitutive equation

$$\psi = \psi(u_{i,A}, u_{iB_1 \dots B_\beta}, u_{iB_1 \dots B_\beta, A}). \tag{4.4}$$

We define stability of the null solution to equations (2.18) - (2.21) in the normal way as follows: given $\varepsilon > 0$, there exist positive definite measure ρ^o and ρ with

$$\rho^o = \rho^o(u_i(0), \dot{u}_i(0), u_{iA_1 \dots A_\alpha}(0), \dot{u}_{iA_1 \dots A_\alpha}(0)), \tag{4.5}$$

$$\rho(t) = \rho(u_i(t), \dot{u}_i(t), u_{iA_1 \dots A_\alpha}(t), \dot{u}_{iA_1 \dots A_\alpha}(t)), \tag{4.6}$$

and scalar $\delta > 0$ such that $\rho^o < \delta$ implies $\rho(t) < \varepsilon$ for all $t \geqslant 0$. We shall prove stability for certain measures ρ^o and ρ by use of the energy functional which, being non-increasing, acts as a natural Liapounov function for the problem.

If we take as initial measure

$$\rho^o = \sup_{\underset{\sim}{X} \varepsilon B} \{\dot{u}_i \dot{u}_i + \sum_{\alpha=1}^{\nu} \dot{u}_{iA_1 \dots A_\alpha} \dot{u}_{iA_1 \dots A_\alpha} + \psi\} \tag{4.7}$$

or, more weakly,

$$\rho^{o} = \int_{B} \rho_{o} [\dot{u}_i \dot{u}_i + \sum_{\alpha=1}^{\nu} \dot{u}_{iA_1 \ldots A_\alpha} \dot{u}_{iA_1 \ldots A_\alpha} + \psi] dV, \tag{4.8}$$

then provided the inertia tensor is positive definite and satisfies a bound of the form

$$| Y_{A_1 \ldots A_\alpha : B_1 \ldots B_\beta} | \leqslant M \tag{4.9}$$

for all α, β such that $1 \leqslant \alpha, \beta \leqslant \nu$, and provided the free energy ψ is positive definite, then ρ^{o} is positive definite and we have

$$E(t) \equiv E(0) \leqslant A \rho^{o} \tag{4.10}$$

for some positive constant A.

Let us next suppose there exist positive constants C or D such that either of the following restrictions on the free energy ψ is satisfied

$$\int_{B} \rho_{o} \psi dV \geqslant C\{\int_{B} [u_{i,A} u_{i,A} + \sum_{\alpha=1}^{\nu} u_{iA_1 \ldots A_\alpha ,C} u_{iA_1 \ldots A_\alpha ,C}] dV\} \tag{4.11}$$

$$\text{or } \int_{B} \rho_{o} \psi dV \geqslant D\{\int_{B} [u_{i,A} u_{i,A} + \sum_{\alpha=1}^{\nu} u_{iA_1 \ldots A_\alpha} u_{iA_1 \ldots A_\alpha}] dV\} \tag{4.12}$$

for all $u_{i,A}$, $u_{iA_1 \ldots A_\alpha}$, $u_{iA_1 \ldots A_\alpha ,C}$. Both these inequalities express the positive definiteness of the free energy, and we note that the first term on the right hand side of each inequality is the term normally used in classical monopolar elasticity. By Poincaré's inequality, it is now easy to show that the positive measure $\rho(t)$ defined by

$$\rho(t) = \int_{B} [u_i u_i + \sum_{\alpha=1}^{\nu} u_{iA_1 \ldots A_\alpha} u_{iA_1 \ldots A_\alpha}] dV \tag{4.13}$$

satisfies

$$\rho(t) \leqslant B \int_{B} \rho_{o} \psi dV \leqslant B E(t) \tag{4.14}$$

for some positive constant B.

It follows that if ρ^{o} is small, then $\rho(t)$ is small for all positive time and we have stability with respect to the measures ρ and ρ^{o}.

Further stability theorems can be proved with respect to different choices of measures but inequalities of the type (4.11), (4.12) will usually be needed. Stronger theorems can be proved in the special case in which the multipolar displacements are gradients of the monopolar displacement, i.e.

$$u_{iA_1 \ldots A_\alpha} = u_{i,A_1 \ldots A_\alpha}. \tag{4.15}$$

We illustrate this by considering a dipolar elastic solid, that is one in which the free energy depends only upon the first and second displacement gradients, thus

$$\psi = \psi(u_{i,A}, u_{i,AB}). \tag{4.16}$$

We investigate the stability of the null solution of the equations (2.25) - (2.27) subject to the following boundary conditions

$$u_i = 0 \quad \text{on } \partial B_1 \ , \ u_{i,B} = 0 \quad \text{on } \partial B_3, \tag{4.17}$$

$$\pi_{Ki}N_K = 0 \text{ on } \partial B_2, \ \pi_{KiB}N_K = 0 \quad \text{on } \partial B_4 \tag{4.18}$$

where ∂B_1 and ∂B_3 are of positive measure, if non-empty, and

$$\partial B_1 \cup \partial B_2 = \partial B, \ \partial B_3 \cup \partial B_4 = \partial B, \tag{4.19}$$

$$\partial B_1 \cap \partial B_2 = \phi \ , \ \partial B_3 \cap \partial B_4 = \phi . \tag{4.20}$$

Further, if $\partial B_1 = \phi$, then to eliminate rigid body translations we take

$$\int_B u_i \, dV = 0, \tag{4.21}$$

and if $\partial B_3 = \phi$, to eliminate rigid body rotations we take

$$\int_B u_{i,B}\, dV = 0. \qquad (4.22)$$

We have, from Sobolev's embedding theorem, that in $\mathbb{R}^3$, $W^{2,2}$ is embedded in C^0 and thus we have an inequality of the form

$$\sup_{\underset{\sim}{X}\varepsilon B}\{u_i u_i\} \leq A_1\{\int_B u_i u_i dV + \int_B u_{i,A} u_{i,A} dV + \int_B u_{i,AB} u_{i,AB} dV\} (4.2$$

for some positive constant A_1. We can also use Poincaré's inequality twice to show that there exists positive constants A_2 and A_3 such that

$$\int_B u_i u_i\, dV \leq A_2 \int_B u_{i,A} u_{i,A}\, dV \leq A_3 \int_B u_{i,AB} u_{i,AB}\, dV. \qquad (4.24)$$

It follows that we have an inequality of the form

$$\sup_{\underset{\sim}{X}\varepsilon B}\{u_i u_i\} \leq A_4 \int_B u_{i,AB} u_{i,AB}\, dV \qquad (4.25)$$

for some positive constant A_4.

Using inequality (4.10) and the conservation of energy it follows that, provided that free energy satisfies an inequality of the form

$$\int_B \rho_o\, \psi\, dV \geq A_5 \int_B u_{i,AB} u_{i,AB}\, dV \qquad (4.26)$$

for some positive constant A_5, we have established stability with respect to the uniform norms ρ and ρ^o defined by

$$(t) = \sup_{\underset{\sim}{X}\varepsilon B}\{u_i u_i\} + \sup_{\underset{\sim}{X}\varepsilon B}\{\dot{u}_i \dot{u}_i\} \qquad (4.27)$$

$$\rho^o = \sup_{\underset{\sim}{X}\varepsilon B}\{\dot{u}_i(0)\dot{u}_i(0) + \dot{u}_{i,A}(0)\dot{u}_{i,A}(0) + \dot{u}_{i,AB}(0)\dot{u}_{i,AB}(0)+\psi\}. (4.2$$

It should be remarked that we are still assuming that $Y_{A_1 \ldots A_\alpha : B_1 \ldots B_\beta}$ is positive definite and that it satisfies a bound of the form (4.9).

A stability analysis based upon a uniform norm of the type (4.27) fails in classical elasticity theory because of the so-called 'focussing effect'. This effect is ruled out in the dipolar solid because of the dependence of the free energy upon the second displacement gradients. Koiter [6], in a discussion of the energy criterion for stability of monopolar elastic bodies, has previously considered the effect of introducing second displacement gradients into the free energy functional, but he studied only a special sub-case of the class of materials considered here.

5. Instability.

We now proceed to find conditions on the free energy ψ of a multipolar elastic material for which the body is unstable. We shall prove instability in the sense of Lagrange, by showing that certain norms of the displacements become formally unbounded in finite time for certain prescribed initial displacements and velocities. In order to do this we modify the concavity technique of Knops, Levine and Payne [3].

We shall examine the instability of the null solution to equations (2.18) - (2.21) subject to boundary conditions of the form (3.5) - (3.9). Firstly, consider the measure F(t) defined by

$$F(t) = \int_B \rho_o [u_i u_i + \sum_{\alpha=1}^{\nu} \sum_{\beta=1}^{\nu} Y_{A_1 \ldots A_\alpha : B_1 \ldots B_\beta} u_{iA_1 \ldots A_\alpha} u_{iB_1 \ldots B_\beta}] dV + \beta (t+t_o)^2 \tag{5.1}$$

where we are assuming that $Y_{A_1 \ldots A_\alpha : B_1 \ldots B_\beta}$ is a positive definite form. We note that F(t) can be considered to be a modified L_2-norm of the displacements.

We can compute the first and second time derivatives of F(t) to obtain

$$\dot{F}(t) = 2\int_{B} \rho_o[u_i\dot{u}_i + \sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta} u_{iA_1\ldots A_\alpha}\dot{u}_{iB_1\ldots B_\beta}]dV + 2\beta(t+t_o), \tag{5.2}$$

and

$$\ddot{F}(t) = 2\int_{B}\rho_o[\dot{u}_i\dot{u}_i + \sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\dot{u}_{iA_1\ldots A_\alpha}\dot{u}_{iB_1\ldots B_\beta}]dV$$
$$+ 2\int_{B}\rho_o[u_i\ddot{u}_i + \sum_{\alpha=1}^{\nu}\sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta} u_{iA_1\ldots A_\alpha}\ddot{u}_{iB_1\ldots B_\beta}]dV + 2\beta. \tag{5.3}$$

Substituting in (5.3) for $\ddot{u}_i$ and $\sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\ddot{u}_{iB_1\ldots B_\beta}$ from the equations of motion (2.18) - (2.21), and using the conservation of energy equation (3.10), we have

$$\ddot{F}(t) = 4(1+\alpha)\int_{B}\rho_o[\dot{u}_i\dot{u}_i + \sum_{\alpha=1}^{\nu}\sum_{\alpha=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta}\dot{u}_{iA_1\ldots A_\alpha}\dot{u}_{iB_1\ldots B_\beta}]dV$$
$$-4(1+2\alpha)E(0)+2\beta+4(1+2\alpha)\int_{B}\rho_o\psi\, dV - 2\int_{B}\rho_o[\frac{\partial\psi}{\partial u_{i,A}}u_{i,A} + \sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta}}$$
$$u_{iB_1\ldots B_\beta} + \sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta,A}}u_{iB_1\ldots B_\beta,A}]\, dV, \tag{5.4}$$

for any positive scalar α.

It then follows, upon using Schwarz's inequality, that

$$F(t)\ddot{F}(t)-(1+\alpha)\dot{F}(t)^2 \geqslant -2(1+2\alpha)[2E(0)+\beta]\,F(t)+4(1+2\alpha)\int_{B}\rho_o\psi\, dV\, F(t)$$
$$-2\int_{B}\rho_o[\frac{\partial\psi}{\partial u_{i,A}}u_{i,A} + \sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta}}u_{iB_1\ldots B_\beta} + \sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta,A}}u_{iB_1\ldots B_\beta,A}]dV\, F(t) \tag{5.5}$$

Thus, if for some positive scalar α, the free energy ψ satisfies an inequality of the form

$$2(1+2\alpha)\int_{B}\rho_o\psi\, dV \geqslant \int_{B}\rho_o[\frac{\partial\psi}{\partial u_{i,A}}u_{i,A} + \sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta}}u_{iB_1\ldots B_\beta}$$
$$+\sum_{\beta=1}^{\nu}\frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta,A}}u_{iB_1\ldots B_\beta,A}]\, dV \tag{5.6}$$

then

$$F(t)\ddot{F}(t) - (1+\alpha)\dot{F}(t)^2 \geqslant -4(1+2\alpha)E(0)F(t) \qquad (5.7)$$

and we can use the results of Knops, Levine and Payne [3] to show that F(t) becomes unbounded in finite time for initial conditions which satisfy $E(0)<0$.

We have thus obtained sufficient conditions for the null solution to be unstable with respect to the measure F(t), but we cannot deduce that the body is not stable with respect to different measures, such as G(t) defined by

$$G(t) = \int_B \rho_o u_i u_i \, dV + \beta(t+t_o)^2 \qquad (5.8)$$

or $F(t) - G(t)$. In fact we can find restrictions on the free energy ψ, different from (5.6), under which G(t) or $F(t)-G(t)+\beta(t+t_o)^2$ do become unbounded in finite time for negative initial energy E(0) by proceeding as follows: from above we can calculate the first and second derivatives of G(t) to be

$$\dot{G}(t) = 2\int_B \rho_o u_i \dot{u}_i \, dV + 2\beta(t+t_o) \qquad (5.9)$$

$$\ddot{G}(t) = 2\int_B \rho_o \dot{u}_i \dot{u}_i \, dV + 2\int_B \rho_o u_i \ddot{u}_i \, dV + 2\beta \qquad (5.10)$$

and then substituting from the equations of motion (2.18)-(2.21) and the energy balance equation (3.10), we have

$$\ddot{G}(t) = 4(1+\alpha)\int_B \rho_o \dot{u}_i \dot{u}_i dV + 2(1+2\alpha)\int_B \rho_o Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta}\, \dot{u}_{iA_1\ldots A_\alpha}\, \dot{u}_{iB_1\ldots B_\beta} dV$$
$$-4(1+2\alpha)E(0) + 2\beta + 4(1+2\alpha)\int_B \rho_o \psi \, dV - 2\int \rho_o \frac{\partial\psi}{\partial u_{i,A}} u_{i,A} \, dV. \qquad (5.11)$$

Hence, it follows from the positive definiteness of $Y_{A_1\ldots A_\alpha : B_1\ldots B_\beta}$, that if the free energy is subject to the restriction

$$2(1+2\alpha)\int_B \rho_o\psi \, dV \geq \int_B \rho_o \frac{\partial\psi}{\partial u_{i,A}} u_{i,A} \, dV \tag{5.12}$$

for some positive scalar α, then

$$G(t)\ddot{G}(t)-(1+\alpha)\dot{G}(t)^2 \geq -2(1+2\alpha)[2E(0)+\beta]G(t) \tag{5.13}$$

and again from Knops, Levine and Payne [3], it follows that, for initial conditions for which the initial energy E(0) is negative, G(t) becomes unbounded in finite time.

Similarly we can show that the null solution is unstable with respect to the measure H(t) defined by

$$H(t)=F(t)-G(t)+\beta(t+t_o)^2 = \int_B \rho_o \sum_{\alpha=1}^{\nu} \sum_{\beta=1}^{\nu} Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta} u_{iA_1\ldots A_\alpha} u_{iB_1\ldots B_\beta} dV + \beta(t+t_o)^2 \tag{5.14}$$

if there exists a positive scalar α such that

$$2(1+2\alpha)\int_B \rho_o \psi dV \geq \int_B \rho_o [\sum_{\beta=1}^{\nu} \frac{\partial\psi}{\partial u_{iB_1\ldots B_\beta}} u_{iB_1\ldots B_\beta} + \sum_{\beta=1}^{\nu} \frac{\partial\psi}{u_{iB_1\ldots B_\beta,A}} u_{iB_1\ldots B_\beta,A}] \, dV \tag{5.15}$$

and provided E(0) can be chosen negative.

Further, in the special case in which

$$Y_{A_1\ldots A_\alpha:B_1\ldots B_\beta} = 1 \text{ if } \alpha=\beta, A_1=B_1,\ldots,A_\alpha=B_\alpha, \tag{5.16}$$
$$= 0 \text{ otherwise},$$

so that

$$F(t)= \int_B \rho_o [u_i u_i + \sum_{\alpha=1}^{\nu} u_{iA_1\ldots A_\alpha} u_{iA_1\ldots A_\alpha}] \, dV, \tag{5.17}$$

it can be similarly shown that the body is unstable with respect to the measure L(t) defined by

$$L(t) \doteq \int_B \rho_o \, u_{iB_1 \ldots B_\beta} u_{iB_1 \ldots B_\beta} \, dV, \ \beta \text{ fixed}, \tag{5.18}$$

provided there exists a positive scalar α such that

$$2(1+2\alpha) \int_B \rho_o \psi dV \geqslant \int_B \rho_o \left[\frac{\partial \psi}{\partial u_{iB_1 \ldots B_\beta}} u_{iB_1 \ldots B_\beta} + \frac{\partial \psi}{\partial u_{iB_1 \ldots B_\beta, A}} u_{iB_1 \ldots B_\beta, A} \right] dV \tag{5.19}$$

and provided $E(0)$ can be chosen negative.

We note that in order to use the results of Knops, Levine and Payne to establish unboundedness in finite time of the measures $F(t)$, $G(t)$, $H(t)$ and $L(t)$ it is not necessary, in fact, to assume that the initial energy $E(0)$ is negative, but that some results can be obtained in the cases when $E(0)=0$ and $E(0)>0$. We refer the reader to the paper [3] of Knops, Levine and Payne for details.

We also note that our instability analysis relies heavily on inequalities of the form (5.6), (5.12), (5.15) and (5.19). An interpretation of inequalities of this type with reference to the existence of potential wells for the free energy function ψ is to be given in a forthcoming paper by Knops, Payne and Wilkes [4].

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