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ISOMORPHICALLY ISOMETRIC PROBABILISTIC NORMED LINEAR SPACES

by

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1. Introduction.

Probabilistic normed linear spaces (briefly PNL spaces) were first studied by A. N. Šerstnev in [1]. His definition was motivated by the definition of probabilistic metric spaces (PM spaces) which were introduced by K. Menger and subsequently developed by A. Wald, B. Schweizer, A. Sklar and others.

In a previous paper [2] we studied the relationship between two important classes of PM spaces, namely, E-spaces and pseudo-metrically generated PM spaces. We showed that a PM space is pseudo-metrically generated if and only if it is isometric to an E-space. In this paper we extend these ideas and results to PNL spaces. We define E-norm spaces and pseudo-norm-generated PNL spaces and show that a PNL space is pseudo-norm-generated if and only if it is isomorphically isometric to an E-norm space. In order to preserve the algebraic structure in a meaning-ful way we require new constructions which differ considerably from those given in the original PM space setting.

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2. Background and Notation.

Let D^+ denote the set of all nondecreasing left-continuous functions F defined on the reals such that $F(x) = 0$ for $x \leqslant 0$ and $\lim_{x\to\infty} F(x) = 1$. The set D^+ is ordered via $F \geqslant G$ if $F(x) \geqslant G(x)$ for all real x. A function $\tau: D^+ \times D^+ \to D^+$ is called a triangle function if for all F, G, K, L in D^+:

($\tau 1$) $\tau(F,G) = \tau(G,F)$,

($\tau 2$) $\tau(\varepsilon_o, F) = F$ where ε_o is the function in D^+ such that $\varepsilon_o(x) = 1$ for $x > 0$,

($\tau 3$) $\tau(F,G) \geqslant \tau(K,L)$ if $F \geqslant K$ and $G \geqslant L$,

($\tau 4$) $\tau(F, \tau(G,K)) = \tau(\tau(F,G),K)$.

In this paper the only particular triangle function needed is the function $\tau_m: D^+ \times D^+ \to D^+$ defined via

$$\tau_m(F,G)(x) = \sup_{t\in[0,1]} \text{Max}(F(tx) + G((1-t)x)-1,0)$$

for each real x.

<u>Definition 2.1.</u> A PNL space is an ordered triple $(L,\mathcal{G},\tau)$ where L is a linear space over the field Λ of complex or real numbers, $\mathcal{G}$ is a mapping from L to D^+ (we often write G_p for $\mathcal{G}(p)$ and $G_p(x)$ for $\mathcal{G}(p;x)$) and τ is a triangle function. In addition, the following axioms are satisfied:

I. $G_p = \varepsilon_o$ if and only if $p = \theta$ (θ is the zero element of L),

II. $G_{ap}(x) = G_p(x/|a|)$ for every p in L and every non-zero a in Λ,

III. $G_{p+q} \geqslant \tau(G_p, G_q)$ for every p, q in L.

3. E-Norm Spaces and Pseudo-Norm-Generated Spaces.

Definition 3.1. Let $(V,||\cdot||)$ be a normed linear space over Λ. The ordered pair $(L,\mathcal{G})$ is a E-norm space over $(V,||\cdot||)$ if and only if the elements of L are functions from a probability space $(\Omega,\mathcal{Q},P)$ into V such that

(a) L is a linear space over Λ under pointwise addition and scalar multiplication where the zero vector, $\hat{\Theta}$, is the constant function of value Θ, the zero vector in V,

(b) For each p in L and each x in R, $\{\omega\in\Omega : ||p(\omega)||<x\}\in\mathcal{Q}$, and

(c) For each p in L, $\mathcal{G}(p)$ is defined via:

$$G_p(x) = P\{\omega\in\Omega : ||p(\omega)||< x\}, \text{ for each } x \text{ in } R.$$

As usual, the functions in L which differ at most on a set of P-measure zero are identified.

Theorem 3.1. If $(L,\mathcal{G})$ is an E-norm space, then $(L,\mathcal{G},\tau_m)$ is a PNL space.

The proof of this theorem is omitted since it is routine and parallels the proof given in [2] of the fact that each E-space is a PM space.

In [3] R. R. Stevens introduced an important class of PM spaces, the metrically generated spaces. In the realm of PNL spaces, the spaces of the next definition play the analogous role.

Definition 3.2. Let L be a linear space over Λ. The ordered pair $(L,\mathcal{G})$ is a norm-generated space if and only if there is a probability space $(\Omega,\mathcal{Q},P)$ and a family $\{||\cdot||_\omega : \omega\in\Omega\}$ of norms for L such that for each $x\in R$ and p in L,

(1) $\{\omega\in\Omega: ||p||_\omega<x\}\in\mathcal{A}$, and

(2) $G_p(x) = P\{\omega\in\Omega: ||p||_\omega< x\}$.

The proof of the following theorem is also straightforward and is similar to the proof given in [3] of the fact that each metrically generated space is a PM space.

Theorem 3.2. If $(L,\mathcal{G})$ is a norm-generated space, then $(L,\mathcal{G},\tau_m)$ is a PNL space.

In the proof ot Theorem 3.2 the fact for each $\omega\in\Omega$, $||\cdot||_\omega$ is a norm as opposed to a pseudo-norms is needed in order to prove that $G_p=\varepsilon_o$ implies that p is the zero-vector. For the purposes of this paper we need, on the one hand, to relax this requirement that each $||\cdot||_\omega$ be a norm. On the other hand, we also wish to avoid irrelevant complications (of having to deal with equivalence classes) which arise when each $||\cdot||_\omega$ in Definition 3.2 is merely assumed to be a pseudo-norm. We obtain the necessary relaxation and sidestep the irrelevant complications by restricting our attention to those spaces which are generated by pseudo-norms in the manner of Definition 3.2 and which at the same time satisfy all the conditions of a PNL space, in particular (I).

Definition 3.3. A PNL space $(L,\mathcal{G},\tau)$ is said to be pseudo-norm-generated if and only if there is a probability space $(\Omega,\mathcal{A},P)$ and a family $\{||\cdot||_\omega:\omega\in\Omega\}$ of pseudo-norms for L such that for each x in R and each p in L, (1) and (2) of Definition 3.2 are satisfied.

Theorem 3.3. A PNL space is pseudo-norm generated if and only if it is isomorphically isometric to an E-norm space.

Proof: First suppose that $(L,\mathcal{G})$ is an E-norm space over the normed linear space $(V,||\cdot||)$. Let $(\Omega,\mathcal{A},P)$ denote the probability space on which the elements of L are defined. For each $\omega\in\Omega$, let $||\cdot||_\omega$ be defined on L via $||p||_\omega = ||p(\omega)||$ for each p in L. It is readily verified that, for each $\omega\in\Omega$, $||\cdot||_\omega$ is a pseudo-norm on L and that the collection $\{||\cdot||_\omega : \omega\in\Omega\}$ satisfies (1) and (2) of Definition 3.2 for each x in R and each p in L. Thus $(L,\mathcal{G})$ is pseudo-norm-generated.

Now suppose π is an isometry from the E-norm space $(L,\mathcal{G})$ onto $(L',\mathcal{G}')$ such that π is also an isomorphism. Let p' in L' denote the value of π at any point p in L and, for any pseudo-norm $||\cdot||_\omega$ on L, let $||\cdot||'_\omega$ be the pseudo-norm on L' defined by $||p'||'_\omega = ||p||_\omega$. By the argument above, $(L,\mathcal{G})$ is pseudo-norm-generated so that, for any p' in L' and any real x,

$$\mathcal{G}'(p';x) = \mathcal{G}(p;x) = P\{\omega\in\Omega : ||p||_\omega < x\} = P\{\omega\in\Omega : ||p'||'_\omega < x\}.$$

Thus it is clear that $(L',\mathcal{G}')$ is pseudo-norm-generated, and half of the proof is complete.

In the other direction, let $(L,\mathcal{G})$ be a PNL space which is pseudo-norm-generated. Then there is a probability space $(\Omega,\mathcal{Q},P)$ and a family $\{||\cdot||_\omega : \omega\in\Omega\}$ of pseudo-norms for L such that for each p in L and each x in R, $\mathcal{G}(p;x) = P\{\omega\in\Omega : ||p||_\omega < x\}$. We desire a pseudo-normed linear space $(V,||\cdot||)$ which contains a 'copy' of each $(L,||\cdot||_\omega)$. We take V to be that subset of the Cartesian product of the family $\{(L,||\cdot||_\omega) : \omega\in\Omega\}$ of pseudo-normed linear spaces which consists of all those elements having only a finite number of non-zero components, i.e.,

$$V = \{f : f \text{ is a function from } \Omega \text{ into L and } f^{-1}(L-\{\Theta\}) \text{ is finite}\}.$$

Addition and scalar multiplication on V are defined pointwise. Then $(V,+,\cdot)$ is a linear space. Define $||\cdot||$ on V via:

$$||f|| = \sum_{\alpha\in f^{-1}(L-\{\Theta\})} ||f(\alpha)||_\alpha$$

for each f in V. It is easy to show that $||\cdot||$ is a pseudo-norm on V. Define an equivalence relation, $\equiv$, on V via $f \equiv g$ if and only if $||f-g|| = 0$. Let V^* denote the collection of equivalence classes in the usual manner; and let f^* denote the equivalence class determined by f. Define $||\cdot||^*$ on V^* via $||f^*||^* = ||f||$ for $f \in f^*$. Then $(V^*, ||\cdot||^*)$ is a normed linear space.

For each p in L let Π_p be the mapping from Ω into V defined as follows:

For each ω in Ω, $\Pi_p(\omega)$ is the function from $\Omega \to L$ whose value $\Pi_p(\omega)(\alpha)$ at each α in Ω is given by

$$\Pi_p(\omega)(\alpha) = \begin{cases} p, & \text{if } \alpha = \omega, \\ \Theta, & \text{if } \alpha \neq \omega. \end{cases}$$

For each p in L let Π_p^* be the mapping from Ω into V^* defined by

$$\Pi_p^*(\omega) = (\Pi_p(\omega))^*,$$

for each ω in Ω. Let p and q be in L. Then $\Pi_p^* = \Pi_q^*$ if and only if for each ω in Ω, $(\Pi_p(\omega))^* = \Pi_p^*(\omega) = \Pi_q^*(\omega) = (\Pi_q(\omega))^*$ which is true if and only if for each ω in Ω, $||\Pi_p(\omega) - \Pi_q(\omega)|| = 0$ which in turn is true if and only if for each ω in Ω, $||p-q||_\omega = 0$. But, for each ω in Ω, $||p-q||_\omega = 0$ implies that for every $x > 0$, $G_{p-q}(x) = 1$ which implies that $p-q = \Theta$, i.e., $p = q$. On the other hand, if $p = q$ then $p - q = \Theta$ so that for each ω in Ω, $||p-q||_\omega = 0$. Thus, for each p and q in L, $\Pi_p^* = \Pi_q^*$ if and only if $p = q$.

Now let $L^* = \{\Pi_p^* : p \in L\}$. It is clear that the mapping Π^* which takes p to Π_p^* is a one-to-one mapping from L onto L^*. Moreover for any Π_p^* in L^* and any x in R,

$$\begin{aligned} &\{\omega \in \Omega : ||\Pi_p^*(\omega)||^* < x\} \\ &\quad = \{\omega \in \Omega : ||\Pi_p(\omega)|| < x\} \\ &\quad = \{\omega \in \Omega : \sum_{\alpha \in (\Pi_p(\omega))^{-1}(L - \{\Theta\})} ||\Pi_p(\omega)(\alpha)|| < x\} \\ &\quad = \{\omega \in \Omega : ||p||_\omega < x\} \in \mathcal{A}. \end{aligned}$$

Define $\mathcal{G}^*$ via:

$\mathcal{G}^*(\Pi_p^*; x) = P\{\omega\in\Omega: ||\Pi_p^*(\omega)||^* < x\}$ for each Π_p^* in L^* and each x in R.

Notice that $(L^*,\mathcal{G}^*)$ is an E-norm space and the mapping Π^* which takes p to Π_p^* is both an isometry and an isomorphism of $(L,\mathcal{G})$ onto $(L^*,\mathcal{G}^*)$. Thus $(L,\mathcal{G})$ is isomorphically isometric to an E-norm space, and the proof is complete.

The construction used to prove the second half ot Theorem 3.3 is not just a simple modification of the construction used in [3] to prove the corresponding result for PM spaces. The need for an entirely new construction was dictated by the added requirement that the algebraic structure of a PNL space be preserved. It is worth noting that a straighforward modification of the new construction does serve to yield the corresponding result in the PM space setting.

References.

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