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ON SUMS OF DEPENDENT UNIFORMLY DISTRIBUTED RANDOM VARIABLES (*)

by

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ABSTRACT

We study and solve several functional equations which yield necessary and sufficient conditions for the sum of two uniformly distributed random variables to be uniformly distributed.

1. Introduction.

In this paper we solve several functional equations, involving the so-called sigma operations, on the space of uniform probability distribution functions. Similar functional equations for other families of operations were studied in a previous paper [1].

The aim of the present work (motivated by some questions in interval analysis) was to prove that if X and Y are two random variables which are uniformly distributed on the intervals [a,b] and [c,d], respectively, then X + Y is uniformly distributed on [a+c,b+d] (resp.,[Min(a+d,b+c), Max(a+d,b+c)]),

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if and only if $Y=\lambda X$ for some $\lambda>0$ (resp., $\lambda<0$), i.e., if and only if X and Y are linearly dependent. We establish this fact and then consider several other questions in order to determine the dependence relationship of two random variables with uniform distributions when the distribution of their sum is assumed to be uniform.

2. Preliminaries.

Let Δ be the set of one-dimensional probability distribution functions, i.e., non-decreasing functions F from $[-\infty,+\infty]$ into $[0,1]$, left-continuous on $(-\infty,+\infty)$ and such that $F(-\infty)=0$, $F(+\infty)=1$. If X is a random variable (r.v.) defined on a probability space $(\Omega,\mathcal{A},P)$, then its distribution function $F_X(t)=P(\{\omega\in\Omega \mid X(\omega)<t\})$ is in Δ.

Definition 2.1. A (two-dimensional) copula is a two-place function C from $[0,1]\times[0,1]$ into $[0,1]$ satisfying the conditions,

(a) $C(a,0)=C(0,a)=0$, $C(a,1)=C(1,a)=a$,

(b) $C(a,c)-C(a,d)-C(b,c)+C(b,d)\geqslant 0$ for $a\leqslant b$, $c\leqslant d$.

It is easy to show that the functions $\mathrm{Min}(a,b)$, $\mathrm{Prod}(a,b)=a.b$ and $T_m(a,b)=\mathrm{Max}(a+b-1,0)$ are copulas; that a copula is non-decreasing in each place; that $T_m\leqslant C\leqslant \mathrm{Min}$, pointwise, for any copula C; and that for any (a,b), (c,d) in $[0,1]\times[0,1]$ and any copula C,

$$|C(a,b)-C(c,d)|\leqslant |a-c| + |b-d|,$$

whence C satisfies the Lipschitz condition, and thus is continuous.

Definition 2.2. If C is a copula then the sigma-operation σ_C is the binary operation on Δ defined by the two-dimensional Lebesgue-Stieltjes integral vía,

$$\sigma_C(F,G)(x) = \iint_{-\infty \leq u+v<x} d\, C\, (F(u),G(v)). \quad (2.1)$$

For any copula C, σ_C is order-preserving on Δ and admits as unit element the step function ε_o given by,

$$\varepsilon_o\,(x) = \begin{cases} 0 \;, & \text{if } x \leq 0, \\ 1 \;, & \text{if } x > 0. \end{cases} \quad (2.2)$$

The operation σ_{Prod} is convolution. The sigma-operations arise naturally in the study of triangle inequalities for probabilistic metric spaces and in the probabilistic extension of the generalized theory of information [8]. The problem of finding those copulas C for which σ_C is a semigroup operation has been completely solved in [3]. Relations between the sigma-operations and other binary operations on Δ have recently been studied in [6]. Furthermore, copulas and sigma operations are closely related to the study of joint distribution functions and the addition of dependent random variables [9]. Regarding these matters, we shall need the following results [9]:

Theorem 2.1. If X,Y are r.v. with distribution functions F_X and F_Y, respectively, and joint distribution function H_{XY}, then there exists a copula C_{XY} (called a connecting copula of X and Y) such that, for all u,v,

$$H_{XY}\,(u,v) = C_{XY}\,(F_X(u),F_Y(v)). \quad (2.3)$$

If F_X and F_Y are continuous then C_{XY} is unique; otherwise C_{XY} is uniquely determined on (Ran F_X) x (Ran F_Y). In the other direction, if F_X, F_Y are in Δ and C is copula, then the function H defined by (2.3) is a two-dimensional distribution function whose margins are F_X and F_Y.

Corollary 2.1. If X,Y are r.v. as above, then the distribution function of X+Y is given by $F_{X+Y}=\sigma_{C_{XY}}(F_X,F_Y)$.

Theorem 2.2. Each of the r.v. X,Y is an increasing functions of the other if and only if their connecting copula is Min. If F_X,F_Y are continuous, then X=Y a.e., if and only if $F_X=F_Y$ and the connecting copula is Min. Moreover if X, Y are both strictly increasing (or strictly decreasing) functions on [0,1] (endowed with Lebesgue measure) then their unique connecting copula is Min.

Theorem 2.3. Let X,Y be r.v. with continuous distributions F_X,F_Y, respectively. Then one of X,Y is a decreasing function of the other a.e., if and only if their connecting copula is T_m. Thus X=Y=c (a.e., c constant), if and only if $F_X(t)+F_Y(c-t)=1$ for all t in R and their connecting copula is T_m. If X is strictly increasing and Y strictly decreasing on [0,1] (endowed with Lebesgue measure) then their unique connecting copula is T_m.

3. Sigma operations and uniform distributions.

In the following we will denote the uniform distribution function on the interval [x,y] by U_{xy} (or $U_{x,y}$), so that, for x<y,

$$U_{xy}(t) = \begin{cases} 0 & , \text{ if } t\leqslant x, \\ \frac{t-x}{y-x} & , \text{ if } x\leqslant t\leqslant y, \\ 1 & , \text{ if } t\geqslant y. \end{cases}$$

We set $U_{xx} = \varepsilon_x$, where $\varepsilon_x(t) = \varepsilon_0(t-x)$, and ε_0 is given by (2.2). Note that

$$U_{xy}(t) = U_{01}\left(\frac{t-x}{y-x}\right).$$

Lemma 3.1. If C is a copula then,

(i) $\sigma_C(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b}$, for any a,b in R;

(ii) $\sigma_C(\varepsilon_a, U_{bc}) = U_{a+b,a+c}$, for any a,b,c in R with b<c;

(iii) If $\sigma_C(U_{ab}, U_{cd}) = U_{ef}$ for some $e \leqslant f$ where a<b and c<d, then $a+c \leqslant e \leqslant f \leqslant b+d$.

In order to study the behavior of sigma-operations on uniform distribution functions we begin with the following basic result.

Theorem 3.1. Let a<b, c<d and let C be a copula. If $\sigma_C(U_{ab}, U_{cd}) = U_{ef}$, for some $e \leqslant f$, then,

(i) $e+f=a+b+c+d$;

(ii) $a+c \leqslant e \leqslant \text{Min}(a+d, b+c) \leqslant \text{Max}(a+d, b+c) \leqslant f \leqslant b+d$;

(iii) $C(u,v)=T_m(u,v)$ for any (u,v) such that either $(b-a)u+(d-c)v \leqslant e-a-c$ or $(b-a)u+(d-c)v \geqslant f-a-c$.

Proof. Suppose $\sigma_C(U_{ab}, U_{cd}) = U_{ef}$ with $e \leqslant f$. Let X,Y be random variables defined on a common probability space, with uniform distribution functions U_{ab}, U_{cd}, respectively. By hypothesis, the distribution function of the sum X+Y is the uniform distribution function U_{ef}. Consequently the linearly of the expectation yields,

$$\frac{e+f}{2} = E(X+Y) = E(X) + E(Y) = \frac{a+b}{2} + \frac{c+d}{2}.$$

This proves (i). Next, using Theorem 3 of [6] we have

$$U_{ef} = \sigma_C (U_{ab}, U_{cd}) \geqslant \tau_{T_m} (U_{ab}, U_{cd})$$

and $\tau_{T_m} (U_{ab}, U_{cd})(x) = \sup_{u+v=x} T_m(U_{ab}(u), U_{cd}(v)) > 0$ when $x > \mathrm{Min}(a+d, b+c)$. Therefore, $U_{ef}(e) = o$ implies $e \leqslant \mathrm{Min}(a+d, b+c)$ whence, using (i) and Lemma 3.1, (ii) follows.

Next, using (3.1), for any x we have

$$U_{ef}(x) = \iint_{u+v \leqslant x} dC(U_{ab}(u), U_{cd}(v)) = \iint_{(b-a)u+(d-c)v \leqslant x-a-c} d\, C\, (u,v),$$

from this and $U_{ef}(e)=0$ it follows that $C(u,v)=0=T_m(u,v)$ whenever $(b-a)u+(d-c)v \leqslant e-a-c$; and $U_{ef}(f)=1$ implies that the C-measure of the region $A=\{(u,v)\in[0,1]^2 \mid (b-a)u+(d-c)v \geqslant f-a-c\}$ is zero, so that for any $(u,v)\in A$, the C-measure of the rectangle with vertices $(u,v),(1,v),(u,1),(1,1)$ must be zero, whence

$$C(u,v)-C(1,v) - C(u,1) + C(1,1) = o,$$

$$C(u,v) = u + v - 1 = T_m (u,v).$$

This proves (iii).

<u>Corollary 3.1.</u> Let $a<b$ and $c<d$. Then $\sigma_C(U_{ab}, U_{cd}) = \varepsilon_k$, for some k, if and only if $k=a+d=b+c$ and $C=T_m$.

Theorem 3.1 suggests the study of the functional equations:

(I) $\sigma_C (U_{ab}, U_{cd}) = U_{a+c,\ b+d}$,

(II) $\sigma_C (U_{ab}, U_{cd}) = U_{\mathrm{Min}(a+d,b+c),\ \mathrm{Max}(a+d,\ b+c)}$,

where a,b,c,d are given and C is to be found. The next two lemmas show that Min and T_m, respectively, are solutions; and Theorem 3.2 shows that they are the only ones.

Lemma 3.2. If a<b and c<d, then $\sigma_{Min}(U_{ab}, U_{cd})=U_{a+c,b+d}$.

Proof. It is well known [4] that σ_{Min} admits the representation $\sigma_{Min}(F,G) = (F^\wedge + G^\wedge)^\wedge$, where, for any H in Δ, $H^\wedge$ is the quasi-inverse of H given by $H^\wedge(t)=\sup\{x|H(x)<t\}$. In particular $U^\wedge_{ab}(x)=(b-a)x+a$ for $x\epsilon(0,1]$, from which the result is immediate.

Lemma 3.3. If a<b and c<d, then $\sigma_{T_m}(U_{ab}, U_{cd}) =$

$= U_{Min(a+d,b+c),Max(a+d,b+c)}$.

Proof. Let X and Y be the r.v. on [0,1] (endowed with Lebesgue measure) defined by $X(t) = (b-a)t+a$ and $Y(t)= -(d-c)t+d$. Their distribution functions are U_{ab}, U_{cd}, respectively, and by Theorem 2.3 their unique connecting copula is T_m. Thus, by Corollary 2.1, the distribution function of X+Y is $\sigma_{T_m}(U_{ab},U_{cd})$. But $(X+Y)(t)=(b+c-a-d)t+(a+d)$, whence the distribution function of X+Y is $U_{Min(a+d,b+c),Max(a+d,b+c)}$.

Remark. It follows at once from Lemma 3.3 that the operation σ_{T_m} is not continuous, with respect to weak convergence, on Δ. The sequence $\{U_{-n,n}\}$ converges weakly to ε_∞, where $\varepsilon_\infty(x)=0$ for all $x<\infty$, but $\sigma_{T_m}(U_{-n,n},U_{-n,n})=\varepsilon_0$ for every n, whereas $\sigma_{T_m}(\varepsilon_\infty,\varepsilon_\infty)=\varepsilon_\infty$. The situation is analogous to that for σ_{Prod} and σ_{Min}. Continuity or discontinuity of the sigma-operations on the subspace $\Delta^+=\{F\epsilon\Delta|\ F(0)=0\}$ is an open question.

Theorem 3.2. Let a<b, c<d and let C be a copula. Then

(i) $\sigma_C(U_{ab}, U_{cd}) = U_{a+c,b+d}$ if and only if C=Min;

(ii) $\sigma_C(U_{ab}, U_{cd}) = U_{Min(a+d,b+c),Max(a+d,b+c)}$ if and only if $C = T_m$.

Proof. Sufficiency follows from Lemmas 3.2 and 3.3. To prove necessity, let X,Y be r.v., defined on a common probability space, with respective uniform distribution function $F_X = U_{ab}$, $F_Y = U_{cd}$ and unique connecting copula C. If $F_{X+Y} = \sigma_C(U_{ab}, U_{cd}) = U_{a+c.b+d}$, then we have the variance relation

$$\sigma^2(X+Y) = \frac{(b+d-a-c)^2}{12} = \sigma^2(x) + \sigma^2(y) + 2\sigma(x,y) = \frac{(b-a)^2}{12} + \frac{(d-c)^2}{12} + 2\sigma(x,y)$$

whence, the covariance $\sigma(X,Y)$ equals $(b-a)(d-c)/12$, so that the correlation coefficient $\rho(X,Y) = +1$. Thus X=A.Y with A>0 and, by Theorem 2.2, C=Min, i.e., (i) holds. In the case $F_{X+Y} = \sigma_C(U_{ab}, U_{cd}) = U_{Min(a+d,b+c),Max(a+d,b+c)}$ we obtain $\sigma(X,Y) = (b-a)(c-d)/12$ and $\rho(X,Y) = -1$, whence X=B.Y with B<0 and Theorem 2.3 yields $C=T_m$.

Corollary 3.2. If X,Y are two r.v. defined on a common probability space and with respective uniform distribution functions on [a,b] and [c,d], then X+Y is uniformly distributed on [a+c,b+d] (resp.,[Min(a+d,b+c),Max(a+d,b+c)]) if and only if X=A·Y with A>0 (resp., X=B.Y with B<0), i.e., if and only if X and Y are linearly dependent.

In the field of interval analysis [2,5,7], instead of the classical real line, one considers the set of closed intervals $I(R) = \{[a,b];\ a \leqslant b \text{ in } R\}$. Addition in I(R) is defined by the Minkowski sum,

$$(*) \qquad [a,b] \oplus [c,d] = [a+c, b+d].$$

Since the interval $[a,b]$ is often viewed as an uncertain number, it seems reasonable -and has been suggested in the literature- to consider a probabilistic model in which each interval $[a,b]\in I(R)$ is replaced by a uniform probability distribution $U_{ab}\in\Delta$, or, equivalently, by a random variable X_{ab} which is uniformly distributed on $[a,b]$. The preceding results show that the mapping $\emptyset: I(R)\to\Delta$ defined by $\emptyset([a,b])=U_{ab}$ is a morphism between the groupoids $(I(R),\oplus)$ and (Δ,σ_C) if and only if C=Min. Thus the sum $X_{ab}+X_{cd}$ of the uniformly distributed "error" random variables X_{ab} and X_{cd} is uniformly distributed on the in-- terval sum $[a+c,b+d]$ if and only if X_{ab} and X_{cd} are (positively) linearly dependent. It follows that the assumptions: (1) the errors are uniformly distributed, (2) the errors associated with distinct intervals are independent, and (3) errors add in accordance with (*), are inconsistent.

<u>Theorem 3.3.</u> Let $a<b$ and $c<d$. Let C be a copula and let σ_C be associative. Then $\sigma_C(U_{ab},U_{cd})=U_{ef}$ for some $e\leqslant f$, if and only if $e=a+c$, $f=b+d$ and C=Min.

Proof. A remarkable result due to Frank [3] states that σ_C is associative if and only if C=Min, C=Prod or C is an ordinal sum of copulas, each of which is either Min or Prod. On the other hand if we assume that $\sigma_C(U_{ab},U_{cd})=U_{ef}$ for some $e\leqslant f$ then by (iii) of Theorem 3.1, we have $C(u,v)=T_m(u,v)$ whenever $(u,v)\in B$, where

$$B = \{(u,v)\in[0,1]^2 \mid (b-q)u+(d-c)v\leqslant e-a-c \text{ or } (b-a)u+(d-c)v\geqslant f-a-c\}$$

Combine these two statements yields that $B=\{(0,0),(1,1)\}$, whence $e=a+c$ and $f=b+d$. And Theorem 3.2 yields C=Min.

The above theorems are uniqueness theorems. In general, i.e., when $a+c<e<\text{Min}(a+d, b+c)\leqslant\text{Max}(a+d,b+c)<f<b+d$, the functional equation $\sigma_C(U_{ab}, U_{cd}) = U_{ef}$ will have several, non related, solutions. This is shown by the following example.

<u>Example 3.1.</u> Let C_1 and C_2 be the copulas defined by

$$C_1(x,y) = \mathrm{Max}\{T_m(x,y),\ \mathrm{Min}(x,y,\mathrm{Max}(x,y)-1/2)\},$$

and

$$C_2(x,y) = \lambda(\{t\in[0,1] \mid t\leqslant x,\ f(t)\leqslant y\}),$$

where λ denotes the Lebesgue measure on $[0,1]$ and $f:[0,1]\to[0,1]$ is given by

$$f(t) = \begin{cases} t+3/4, & \text{if } 0\leqslant x < 1/8, \\ t+1/2, & \text{if } 1/8\leqslant x < 1/4 \text{ or } 3/8\leqslant x < 1/2, \\ t, & \text{if } 1/4\leqslant x < 3/8, \\ t-1/2, & \text{if } 1/2\leqslant x < 5/8, \\ t-1/4, & \text{if } 5/8\leqslant x < 7/8, \\ t-3/4, & \text{if } 7/8\leqslant x\leqslant 1. \end{cases}$$

C_1 is a commutative copula which is not associative ($1/8 = C_2(11/16,\ C_1(5/8,\ 3/8)) \neq C_1(C_1(11/16,\ 5/8),\ 3/8)=0$) and C_2 is a non-commutative (and therefore non-associative) copula and a straightforward computation shows that

$$\sigma_{C_1}(U_{01},\ U_{01}) = \sigma_{C_2}(U_{01},\ U_{01}) = U_{1/2,\ 3/2}.$$

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