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GEOMETRODYNAMICS OF SOME NON-RELATIVISTIC INCOMPRESSIBLE FLUIDS

by

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1. Introduction.

In some previous papers [1,2] we proposed a geometric formulation of continuum mechanics, where a continuous body is seen as a suitable differentiable fiber bundle C on the Galilean space-time M, beside a differential equation of order k, $E_k(\mathbb{C})$, on $\mathbb{C}$ and the assignement of a frame Ψ on M. This approach allowed us to treat continuum mechanics as an unitary field theory and to consider constitutive and dynamical properties in a more natural way. Further, the particular intrinsic geometrical framework allowed to utilize directly the formal theory of differential equations in order to obtain criteria of existence of solutions.

In the present paper we apply this general theory to some incompressible fluids. The scope is to demonstrate that also for these more simple materials our theory is a suitable tool in order to understand better the fundamental principles of continuum mechanics.

Space-time is the first important geometrical object to study. In classical mechanics space-time (Galilean space-time) is individued by a couple ($\mathfrak{F}$,g), where:

1) F is the fiber bundle structure between affine spaces

$$F \equiv \{\tau : M \to T\}$$

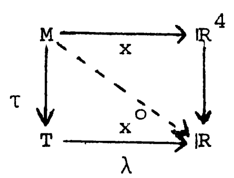
with: (a) M=4-dimensional affine space <u>(space-time)</u>, (the corresponding affine structure is $(M,\underline{M},\alpha)$; (b) T=1-dimensional affine space <u>(time-space)</u>, (the corresponding affine structure is $(T,\underline{T},\beta)$); $\underline{T}$ is supposed oriented; (c) τ= an affine surjective map, of constant rank=1, which associates to each event $p \in M$ its <u>time</u> $\tau(p) \in T$. We write

$$S = \ker(\bar{\bar{D}}\tau) \subset \underline{M}^{\,(*)}.$$

2) g is a field of geometric objects (see ref. [3]) of the bundle of geometric objects $(vS^{o}_{2}M = M \times S^{+} \odot S^{+}, M, \chi; vS^{o}_{2})$ (**) given by $g(p) = (p, \bar{g})$, where $\bar{g}$ is an Euclidean structure on S; g is called <u>metric field.</u>

If an Euclidean structure is taken (i.e. a unit measure for time is chosen) on $\underline{T}$, then $\underline{T}$ is identifiable with $\mathbb{R}$, and we recognize a canonical 1-form $\sigma = d\tau$ on space-time.

Moreover, on M we recognize a very important category of coordinate systems, the so-called <u>adapted coordinate systems</u>, i.e. diffeomorphisms $x : M \to \mathbb{R}^4$ such that the following diagram is commutative:



(*) $\bar{\bar{D}}\tau$ means the Frechet derivation of τ, (see ref. [1,I]).

(**) S^{+} means the vector space dual of S. $\odot$ is the symbol of symmetric tensor product. vS^{o}_{2} means the relative covariant functor which characterizes the fiber bundle $\chi : vS^{o}_{2}M \to M$ as a bundle of geometric objects.

where λ is a coordinate on T. In such cases we get $dx^o = \sigma$ and $\partial x_k(p) \in vTM=MxS$. In other words the coordinate lines $x_{k,p}$, passing for $p \in M$, have values into $M_{\tau(p)}$ only, for k=1,2,3.(*) With respect to the canonical connection on TM, we have $\nabla g = \nabla \sigma = 0$ and for the connection symbols $G^{\alpha}_{\beta\gamma}$ we get in adapted coordinates:

(a) $G^j_{ki} = [ki,s] g^{sj}$ with

$$[ki,s] = \frac{1}{2} [(\partial x_k \cdot g_{is}) + (\partial x_i \cdot g_{sk}) - (\partial x_s \cdot g_{ki})]$$

= Christoffel symbols;

(b) $G^o_{\beta\alpha} = G^o_{\alpha\beta} = 0$.

In order to refer absolute concept to observers, we must introduce a frame, i.e. a map $\Psi : TxM \to M$ such that:

(i) $\Psi_t : M \to M_t = \tau^{-1}(t)$ is a retracting map; (**)

(ii) For each $t, t' \in T$, $\Psi_{t'}(\Psi_t(p)) = \Psi_{t'}(p)$.

With respect to a frame we recognize some important adapted coordinates such that $\partial x_o = \dot{\Psi}$ = the velocity of the frame. In a frame coordinate system we get $G^i_{oj} = G^i_{jo} = 0$.

We assume that any physical entity be represented by a field of geometric objects. This assumption is justified since such structures satisfy the general requirement of "covariance". On the other hand in classical mechanics tensor fields are not sufficient to describe all physical entities (for example the metric field is not a tensor field on M). Then we can enunciate the following important

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(*) Latin indices run from 1 to 3. Greek indices run from 0 to 3.

(**) For the definition of a retracting map see ref. [4].

<u>Theorem 1.1.</u> (REF).(*) A frame induces a canonical projection of any bundle of geometric objects $(W\equiv\mathbb{B}(M),M,\chi_W;\mathbb{B})$ onto two subbundles of W, $\overset{\circ}{W}_\Psi$ and $\hat{W}_\Psi$, called <u>time and space-bundle</u> respectively, associated to W by means of Ψ.

This theorem allows us to give a precise meaning to the concept of observed physical entity. In fact, let $S_\Psi\equiv M/\sim$ be the <u>observed space</u> by means of the frame Ψ, i.e. the space of equivalence classes individuated by the equivalence relation: $p\sim p' \Leftrightarrow \exists t'\in T,\ p'=\Psi_{t'}(p)$. We call <u>observed bundle</u> corresponding to W the bundle $W_\Psi=\mathbb{B}(S_\Psi)$. We recognize a canonical projection, $\chi_\Psi:W\to W_\Psi$ and, if f is a field of geometric objects of χ_W, then the map

$$f_\Psi\equiv\chi_\Psi\circ f\circ j_\Psi^{-1}:T\times S_\Psi\to W_\Psi$$

represents the field of geometric objects corresponding to f, ($j_\Psi:M\to T\times S_\Psi$ is the bijective map induced by Ψ). This analysis applies in particular to tensor fields on M. Thus we get that

$$s_\Psi=[(-1)^r\,\dot{\psi}^{i_1}\ldots\dot{\psi}^{i_r}\,s^{o\ldots o}{}_{j_1\ldots j_s}+(-1)^{r-1}\,\dot{\psi}^{i_2}\ldots\dot{\psi}^{i_r}\,s^{i_1 o\ldots o}{}_{j_1\ldots j_s}+$$
$$+\ldots+(-1)^{r-k}\,\dot{\psi}^{i_{k+1}}\ldots\dot{\psi}^{i_r}\,s^{i_1\ldots i_k o\ldots o}{}_{j_1\ldots j_s}+\ldots+$$
$$+s^{i_1\ldots i_r}{}_{j_1\ldots j_s}]\circ j_\Psi^{-1}\,(\partial x_{i_1})_\Psi\otimes\ldots\otimes(dx^{j_s})_\Psi$$

is the observed tensor field, in adapted coordinates, corresponding to

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(*) In this paper we use some propositions that we have proved in ref. [2]. We write "REF" after the numbers of these propositions.

$$s = s^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s}\, \partial x\alpha_1 \otimes \ldots \otimes dx^{\beta_s}.$$

The <u>Galilean group</u> <u>G</u> is the group of transformations of M which preserve the Galilean space-time structure. Therefore a Galilean transformation is an affine bundle transformation of F, (f, f_T), such that for the corresponding linear maps $\bar{f}$ and $\bar{f}_T$ we have: (a) $\bar{f}|\, S\in O(S)$; (b) $\bar{f}_T = id_{\underline{T}}$.

The set of linear maps such as $\bar{f}$ forms a group: the <u>linear Galilean group</u> <u>GL</u>. <u>G</u> can be seen as a pseudogroup on M also, i.e., the set of solutions of <u>Galilean differential equation</u>, G(M), which in adapted coordinates is given by

$$G(M) \qquad \begin{aligned} &(\partial x_i . f^j)(\partial x_s . f^m) g_{jm} o f = g_{is} \\ &(\partial x_o . f^o) = 1. \\ &(\partial x_k . f^o) = 0. \end{aligned}$$

G(M) admits a related pseudogroup of Lie: the <u>linearized Galilean equation</u>

$$(\partial x_j . g_{is}) X^j + (\partial x_i . X^j) g_{js} + (\partial x_s . X^m) g_{mi} = 0$$

$$(\partial x_\alpha . X^o) = 0$$

whose solutions are <u>infinitesimal Galilean transformations</u>, i.e. Killing vector fields X on M with respect to <u>G</u>. (*)

(*) For more details on the Galilean space-time structure see [2].

2. Geometric structure of perfect incompressible fluids.

A perfect fluid can be considered as a geometric structure on the Galilean space-time. More precisely we give the following

<u>Definition 2.1.</u> A <u>perfect incompressible fluid</u> is a triplet PIF = $(\varphi, E_2(\mathbb{C}), \Psi)$, where:

1) Ψ is a frame on M;

2) $\varphi = (\mathbb{C}, M, \chi_{\mathbb{C}}; \mathbb{C})$ is a bundle of geometric objects on the Galilean space-time, called <u>configuration bundle</u>, such that

$$\mathbb{C} = TM \oplus \Pi \oplus \Theta \cong M \times \underline{M} \oplus \mathbb{R} \oplus \mathbb{R};$$

$TM = M\times\underline{M}$ is the <u>space of velocity</u>, $\Pi = M\times \mathbb{R}$ is the <u>space of pressure</u>, and $\Theta = M\times\mathbb{R}$ is the <u>space of temperature.</u> The covariant functor C is directly individuated be means of the tangent functor T. More precisely, if ν is the category of differentiable manifolds with diffeomorphisms as morphisms, C is the functor $C:\nu\to\nu$ such that for any differentiable manifold V

$$C(V) = TV \oplus T^{o}_{o}M \oplus T^{o}_{o}M$$

and for any diffeomorphism $f:V\to V'$

$$C(f) = T(f) \oplus T^{o}_{o}(f) \oplus T^{o}_{o}(f),$$

i.e. $$C(f)(p,u,\lambda,\mu) = (f(p), \bar{\bar{D}}f(p)(u), \lambda, \mu).$$

Any section of $\chi_{\mathbb{C}}$ is called a <u>configuration</u> of PIF. Thus a configuration is a triplet $c=(v,p,\theta)$ where v is tha <u>velocity field</u>, p is the <u>pressure field</u> and θ is the <u>temperature field</u>.

3) $E_2(\mathbb{C})$ is a second-order differential equation on the configuration bundle, <u>(dynamical equation)</u> given by

$$E_2(\mathbb{C}) = \overset{c}{E}(\mathbb{C}) \cap \overset{m}{E}(\mathbb{C}) \cap \overset{e}{E}(\mathbb{C}) \subset J\mathcal{D}^2(\mathbb{C}), \quad (*)$$

where: (a) $\overset{c}{E}(\mathbb{C})$ is the <u>continuity equation</u> given as the kernel of the differential operator <u>(constitutive map of continuity)</u>

$$\mathcal{Z}.: C^\infty(\mathbb{C}) \to C^\infty(T^o_oM)$$

$$\mathcal{Z}.c = \mathrm{div}(\rho v),$$

where ρ is a constant numerical function on M called <u>mass density.</u>

(b) $\overset{m}{E}(\mathbb{C})$ is the <u>motion equation</u> given as the kernel of the differential operator <u>(constitutive map of motion)</u>

$$\mathcal{D}.: C^\infty(\mathbb{C}) - C^\infty(vTM)$$

$$\mathcal{D}.c = \mathrm{div}(\rho\ v \otimes v + p\ g) - \rho B,$$

where B is the body force field. (c) $\overset{e}{E}(\mathbb{C})$ is the <u>energy equation</u> given as the kernel of the differential operator <u>(constitutive map of energy)</u>

$$\Xi.: C^\infty(\mathbb{C}) \to C^\infty(T^o_oM)$$

$$\Xi.c = \mathrm{div}\{v\rho[e+\tfrac{1}{2}g(v_{\wedge\psi}, v_{\wedge\psi})] \quad -\lambda\ \mathrm{grad}_\psi\theta\ +$$

$$-p\ '\ g(v_{\wedge\psi})\lrcorner g\} - \rho g(B, v_{\wedge\psi}),$$

(*) $J\mathcal{D}^2(\mathbb{C})$ is the jet-derivative space of second order on $\mathbb{C}$, (see ref. [1.II]).

e is the <u>interior energy</u> (corresponding to the configuration c), λ is the <u>thermal conductivity</u> (corresponding to c) and $v_{\hat{\psi}}$ is the <u>space-component</u> of v obtained by means of ψ . 'g denotes the metric isomorphism $vTM = vT^{+}M$.

A <u>dynamical configuration</u> c is a configuration such that it is a solution of the dynamical equation $E_2(\mathbb{C})$, i.e. for any $p \in M$ there exists a neighborhood U such that

$$D^2c : U \to E_2(\mathbb{C}) \subset J\,\mathcal{D}^2(\mathbb{C}).$$

The <u>state bundle</u> is the differentiable fiber bundle

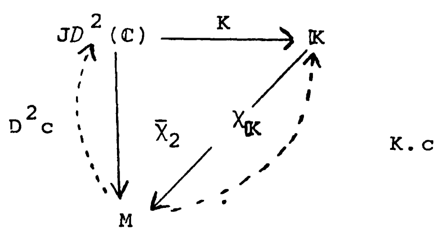
$$\bar{\chi}_2 : J\,\mathcal{D}^2(\mathbb{C}) \to M.$$

A <u>dynamical state</u> is a section s of $\bar{\chi}_2$ which takes values just into $E_2(\mathbb{C})$. The <u>configuration permitted from the state</u> s is the unique configuration c such that $D^2c = s$.

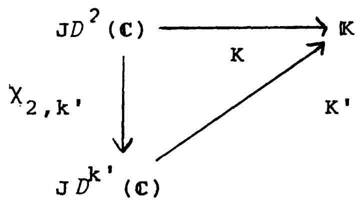
In this framework any constitutive map for a perfect incompressible fluid can be seen as a differential operator of second order on the configuration bundle. Thus, for any configuration c, a constitutive map K can be factorized as

$$K.c = K \circ D^2c,$$

where K is a fiber bundle morphism over M, $K : J\,\mathcal{D}^2(\mathbb{C}) \to \mathbb{K}$, being $(\mathbb{K}, M, \chi_{\mathbb{K}}; \mathcal{K})$ a bundle of geometric objects over M, thus the following diagram is commutative:



We say that a constitutive map of PIF is <u>reducible</u> to one of order $0 \leqslant k' < 2$ if there exists a differential operator of order k', K', such that the following diagram is commutative:



here $\chi_{2,k'}$ is the canonical projection. In these cases we simply say that the constitutive map K is of order k'.

If $\{x^\alpha\}$ is an adapted coordinate system on M we get that a constitutive map is expressible as

$$(K.c)^j(p) = F^j(x^\beta(p), c^i(p), (\partial x_\alpha . c^i)(p)),$$

being $(K.c)^j \equiv y^j o(K.c)$, where $\{y^j\}$ is a coordinate system on $\mathbb{K}$. In tab. I we list some important constitutive maps.

An <u>internal constraint</u> is a sub-bundle over M, $\mathbb{C}'$, of $\mathbb{C}$. For a PIF there is a canonical internal constraint: the <u>velocity constraint</u> $\mathbb{C}^v = V \oplus \Pi + \theta \subset \mathbb{C}$, where $V = M \times I$, being

$$I \equiv \{v \in \underline{M} \mid \langle \underline{\tau}, v \rangle = 1\},$$

$\underline{\tau}$ is the linear map associated to τ.

By exploiting the velocity constraint, we have for $E_2(\mathbb{C})$ the following expression in adapted coordinates:

1) continuity equation : $G^j_{jo} + G^j_{jk} v^k + (\partial x_i . v^i) = 0$

2) motion equation:

$$\rho[G^j_{oo}+2G^j_{ok}v^k+G^j_{ik}v^iv^k+(\partial x_o.v^j)+v^i(\partial x_i.v^j)]-[-pG^i_{ki}g^{kj}+$$

$$-pG^j_{ki}g^{ki}+(\partial x_i.(pg^{ij}))] \quad -\rho B^j = 0$$

3) energy equation:

$$\rho[(\partial x_o.e)+v^k(\partial x_k.e)] + [G^j_{ji}q^i + (\partial x_i.q^i)] -[G^j_{ji}(v^k- \dot{\psi}^k)g_{ks}$$

$$+(\partial x_i.(v^k- \dot{\psi}^k))g_{ks}] \quad pg^{si} = 0.$$

A more semplificate expression is obtained by taking a frame-coordinate system. In fact, in these cases $G^i_{oj}=G^i_{jo}=0 = \dot{\psi}^k$. Furthermore, for inertial frames $G^i_{oo}=0$ and we recover the usual expression for the differential equations governing a perfect incompressible fluid.

Tab.I - Some important reducible constitutive maps for PIF

order	name	$\mathbb{K}$	K.-symbol	K.c-symbol
0	rheological	$vS^2_oM=MxS\ominus S$	$\mathfrak{R}$	$P= -pg$
0	body force	$vTM=MxS$	$\mathfrak{B}$	B
0	power force	T^o_oM	W.	$w=\rho\ g(B,v_{\wedge\dot{\psi}})$
0	thermal conductivity	T^o_oM	Λ.	λ
1	heat flux	vTM	Q.	$q= -\lambda\ grad_\psi\theta$
0	momentum flux	$S^2_oM=Mx\underline{M}\ominus\underline{M}$		$T.c=\rho\ v\otimes v - P$
0	interior energy	T^o_oM	E.	e
1	energy flux	$TM=Mx\underline{M}$	&.	$\epsilon=v\rho[e+\frac{1}{2}g(v_{\wedge\psi},v_{\wedge\psi})] +$ $+q-'g(v_{\wedge\psi}) \lrcorner P$
0	mass flux	TM	μ.	$\mu.c= \rho v$

<u>Remark.</u> For the intrinsic expression of $grad_\psi\theta$ see ref. [2,II] . In adapted coordinates q is given by

$$q = - \lambda(\partial x_i.\theta)g^{ij} \ \partial x_j.$$

3. Symmetry properties for perfect incompressible fluids.

Definition 3.1. A symmetry of PIF is a vector fiber bundle isomorphism $f=(f_{\mathbb{C}}, f_M): \varphi \to \varphi$ such that: (a) $f_M \in \underline{G}$; (b) $J D^2(f_{\mathbb{C}})|E_2(\mathbb{C})$ is a fiber bundle transformation of $E_2(\mathbb{C})$.

The set of symmetries of PIF is a group $G(PIF) \subset Hom(\chi_{\mathbb{C}}, \chi_{\mathbb{C}})$: the symmetry group of PIF. The c-isotropy group, $G_c(PIF)$, is the subgroup of G(PIF) such that the configuration c is f-symmetric.

Then we have the following

Proposition 3.1. (REF) Let a configuration c be given. Any configuration c' f-related to c, with $f \in G(PIF)$, is a dynamical configuration if and only if c is a dynamical configuration also.

Let us now consider the relation between G(PIF) and the constitutive maps of PIF.

Definition 3.2. Let $\kappa = (K, M, \chi_{\mathbb{K}}; \mathcal{K})$ be a vector bundle of geometric objects on M.

1. A physical isomorphism of κ is a vector fiber bundle isomorphism (Ψ, f_M) of $\chi_{\mathbb{K}}$ with $f_M \in \underline{G}$. The set of physical isomorphism of κ is denoted by $F_{\mathbb{K}}$ and called the physical group of κ.

2. A K-symmetry for a constitutive map $K.: C^{\infty}(\mathbb{C}) \to C^{\infty}(\mathbb{K})$ of PIF is an element of $f \in G(PIF)$ such that there exists a vector bundle isomorphism of $\mathbb{K}$, over f_M, such that

$$\Psi \circ K = K \circ J D^2(f_{\mathbb{C}}).$$

K(PIF) is the set of K-symmetries (K-group of PIF).

A <u>Galilean physical structure</u> is a couple (M,k), where: M is the space-time; (b) k is a field of geometric ob-ts of K. The set of all $(\Psi, f_M) \in F_{\mathbb{K}}$ such that k is related itself is called the <u>general k-isotropy group on M</u>: 0_k.

Two important examples of 0_k are:

$0_g \cong \underline{G} \times 0(S)$ and

) $0_\eta = S\underline{G} \times S0(S)$, where η is the <u>volume form</u> on M, i.e. a tion $\eta : M \to v\Lambda_3^o M \equiv M \times S^+ \wedge S^+ \wedge S^+$ such that $\eta(p) = (p, \bar{\eta})$, with olume form on S. $S\underline{G}$ is the special Galilean group (see [2,I]).

<u>Theorem 3.1.</u> (REF) If k' and k are two fields of geome-ic objects of the vector bundle κ, then the following condi-ons are equivalent:

) $k' = h.k$ with $h : M \to \mathbb{R}$ a numerical function invariant for $\in \underline{0}_k$ and $f_M \in \underline{0}_k$, ($\underline{0}_k$ is the canonical image of 0_k into $\underline{G}$);

i) $0_{k'} = 0_k$.

We now make some further important definitions.

<u>Definition 3.3.</u> The <u>general k-isotropy group on M with</u> ght a belonging to a sub-group A of $\mathbb{R}$ is

$$0_{Ak} \equiv \{(\Psi, f_M) \in F_{\mathbb{K}} \mid ak = \Psi \circ k \circ f_M^{-1},\ a \in A\}.$$

important example is when $k \equiv \eta$ and $A \equiv \{-1, +1\}$. We get

$$0_{\{-1,+1\}\eta} \cong \underline{G} \times \mathrm{Unim}(S).$$

ourse $0_\eta \subset 0_g \subset 0_{\{-1,+1\}\eta}$.

<u>Definition 3.4.</u> 1. A dynamical configuration c is isot (PIF) = 0_c.

2. Let us put $K^c(PIF) \equiv \{(f_{\mathbb{C}}, f_M; \Psi) \in K(PIF) \mid K.c = \Psi o K.c \; o \; f_M^{-1}\}$. Let $\underline{G}K^c(PIF)$ and $Hom_M^{K^c}(\chi_{\mathbb{K}}, \chi_{\mathbb{K}})(PIF)$ be the canonical images of $K^c(PIF)$ into $\underline{G}$ and $Hom_M(\chi_{\mathbb{K}}, \chi_{\mathbb{K}})$, respectively. Let us put

$$\bar{F}K^c(PIF) \equiv \underline{G}K^c(PIF) \times Hom_M^{K^c}(\chi_{\mathbb{K}}, \chi_{\mathbb{K}})(PIF).$$

Then the dynamical configuration c is <u>(k,K)-isotropic</u> if

$$FK^c(PIF) = 0_k.$$

Thus we have the following

<u>Proposition 3.2.</u> (REF) If c is (k,K)-isotropic dynamical configuration, then K.c=h.k, with $h: M \to \mathbb{R}$ a $\underline{0}_k$-invariant numerical function.

Of particular importance is the <u>physical group of the stress space</u> $\mathbb{P} \equiv vS_o^2M$, $F_{\mathbb{P}}$. We recall (see ref. [2,II]) that

$$F_{\mathbb{P}} = \underline{G} \times GL(S).$$

Moreover, $F\mathcal{R}^c(PIF)$ coincides with the general g-isotropy group on M: $0_g \cong \underline{G} \times 0(S) \cong F\mathcal{R}^c(PIF)$. Namely, any dynamical configuration for a perfect incompressible fluid is $(g,\mathcal{R})$-isotropic.

We now prove that a perfect incompressible fluid is a "fluid material" in the sense defined in ref. [2.II].

<u>Proposition 3.3.</u> For a perfect incompressible fluid

$$F\mathcal{R}(PIF) \supset 0_{\{-1,+1\}\eta}.$$

<u>Proof.</u> In fact, we have seen that

$$F\mathcal{R}(PIF) \supset F\mathcal{R}^c(PIF) \cong \underline{G} \times 0(S). \qquad (1)$$

On the other hand we know (see ref. [2,II]) that

$$\underline{G} \times GL(S) \subset \mathcal{F}\mathcal{R}(PIF). \qquad (2)$$

Further, Unim(S) is the smallest sub-group of GL(S) which contains O(S). Therefore, from (1) and (2) we conclude that necessarily

$$\mathcal{F}\mathcal{R}(PIF) \supset \underline{G} \times Unim(S) \cong O_{\{-1,+1\}\eta}. \quad \square$$

4. Some considerations on the geometric structure of Newtonian incompressible fluids.

For Newtonian incompressible fluids the situation is a little more complicated than for perfect incompressible fluids.

<u>Definition 4.1.</u> A <u>Newtonian incompressible fluid</u> is a triplet NIF=$(\varphi, E_2(\mathbb{C}), \Psi)$, where:

1. Ψ and φ are as in Definition 2.1.
2. $E_2(\mathbb{C})$ is a second order partial differential equation on φ given by $E_2(\mathbb{C}) = \overset{c}{E}(\mathbb{C}) \cap \overset{m}{E}(\mathbb{C}) \cap \overset{e}{E}(\mathbb{C})$, where: (i) $\overset{c}{E}(\mathbb{C})$ is as in Definition 2.1, (ii) $\overset{m}{E}(\mathbb{C})$ = ker (D), being D a constitutive map given by

$$D.c = \text{div}[\rho\, v \otimes v + pg - 2(\Sigma.c)\, \mathcal{L}g.v] - \rho B, \quad (*)$$

where $\Sigma.$ is a constitutive map of order zero called <u>Newtonian viscosity</u>, (iii) $\overset{e}{E}(\mathbb{C})$=ker.$(\Xi)$, with $\Xi.$ a constitutive map given by

$$\Xi.c = \text{div}\{ v\rho\, [e + \tfrac{1}{2} g(v_{\wedge\psi}, v_{\wedge\psi})] + q - {}'g(v_{\wedge\psi}) \lrcorner [-pg + 2(\Sigma.c)\mathcal{L}g.v] \}$$
$$- \rho g(B, v_{\wedge\psi}).$$

(*) Here $\mathcal{L}g.$ denotes the Lie differential individuated by the metric field g; (see ref. [1,II]).

e rheological map is given by a constitutive map reducible to

$$\mathcal{R}.c = -pg + 2(\Sigma.c)\,\mathcal{L}\,g.v. \qquad (3)$$

Further, we note that a Newtonian incompressible fluid is also a "fluid material". In fact it has the $(g,\mathcal{R})$-isotropic dynamical configuration c_o corresponding to rigid motion $(v=0)$. In such case, we get

$$\mathcal{R}.c_o = -\,pg.$$

Thus, for a Newtonian incompressible fluid we have that

$$\mathcal{F}\mathcal{R}^{c_o}(NIF) \cong \underline{G} \times O(S) \subset \mathcal{F}\mathcal{R}(NIF) \subset \underline{G} \times GL(S),$$

and, as in Proposition 3.3., we get

$$\mathcal{F}\mathcal{R}(NIF) \supset O_{\{-1,+1\}\eta}$$

To any (dynamical) configuration $c=(v,p,\theta)$ we can associate a local flow $\emptyset$ on M such that $\partial\emptyset=v$. consequently, to any configuration c we can associate one and only one deformation gradient $F=D_2\emptyset$, (where D is the symbol of space derivative (see ref. [2,I])) or strain U, (such that $U_\lambda = \emptyset^+_\lambda g, \forall\lambda\in\underline{T}$). Then a Newtonian incompressible fluid can be seen as a <u>simple material.</u> In fact, the constitutive configuration which represents a Newtonian incompressible fluid in the <u>consitutive configuration space</u> (these notations are presented carefully in ref. [2,II]), is individuated by constitutive maps K. which can be factorized as

$$\begin{array}{ccc} C^{\infty}(\mathbb{C}) & \xrightarrow{K.} & C^{\infty}(\mathbb{K}) \\ & \searrow & \nearrow \; \underline{K}. \\ & \Phi(M)\times \underline{C}^{\infty}(\underline{T}\times M,\underline{U})\times C^{\infty}(\Pi\oplus\Theta). & \end{array} \qquad (*)$$

Equivalently

$$K.(v,p,\theta) = \underline{K}.(\emptyset,U,p,\theta). \qquad (4)$$

In fact, in the 6-plet $(\mathfrak{B}.,\hat{R}.,\Theta.,\Lambda.,\mathcal{R}.,E.)$ (**) giving the constitutive configuration of a Newtonian incompressible fluid, the rheological map is the only non-trivial constitu tive map, (i.e., the only map chich is not reducible to one of order zero, constant on the sub-bundle TM of $\mathbb{C}$). But for this map we can just find a map

$$\mathcal{R}.:\Phi(M)\times\underline{C}^{\infty}(\underline{T}\times M,\underline{U})\times C^{\infty}(\Pi\oplus\Theta) \to C^{\infty}(\mathbb{P})$$

which satisfies equation (4). More precisely, we have

$$\mathcal{R}(\emptyset,U,p,\theta) = -pg + 2\chi\{\lim_{\lambda\to 0}\frac{U_{\lambda}-g}{\lambda}\},$$

where χ is the value of the Newtonian viscosity, correspon- ding to (p,θ). (***).

(*) $\underline{U} = vS^{o}_{2}M = M\times S^{+}\theta S^{+}$ is the <u>strain space</u>, $\Phi(M)$ is the space of (local) flows on M and $\underline{C}^{\infty}(\underline{T}\times M,\underline{U})$ is the space of C^{∞}-maps $\underline{T}\times M \to \underline{U}$ which can be defined also on sub-spaces $J\times U$ of $\underline{T}\times M$ only.

(**) $\hat{R}.$ and $\Theta.$ are the <u>mass density</u> and <u>temperature</u> constitutive maps respectively.

(***) A perfect incompressible fluid is a simple material. In fact, in the 6-plet which gives the relative constitutive con- figuration there are constitutive maps of order zero, constant on the subbundle TM of C.

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