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ON ORDER AND BETWEENNESS COMPATIBILITY

by

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A ternary relation B in a non-empty set X is a subset of X x X x X. If B satisfies the axioms

(PB.1) $(a,b,a) \in B \Leftrightarrow a=b$,

(PB.2) $(a,b,c) \in B$ and $(b,d,e) \in B \Rightarrow (c,b,d) \in B$ or $(e,b,a) \in B$;

then B is a <u>partial-betweenness relation</u> in X and (X,B) is a <u>partial-betweenness set</u> ([6]).

If, in addition, B satisfies:

(T.A.) For all $a,b,c \in X$ either $(a,b,c) \in B$ or $(b,c,a) \in B$ or $(c,a,b) \in B$, then B is a <u>total-betweenness relation</u> in X, and (X,B) is a <u>betweenness-set.</u>

If B satisfies (PB.1), (PB.2) and

(O.B.) $(a,a,b) \in B$, $(b,b,c) \in B$ and $(c,c,a) \in B \Rightarrow (a,b,c) \in B$ or $(b,c,a) \in B$, or $(c,a,b) \in B$,

then B is called an <u>order-betweenness relation</u> in X, and (X,B) is an <u>order-betweenness set</u> ([3]).

For example, if $\theta \subset X \times X$ is a partial order, the relation B_0 given by $(x,y,z) \in B_\theta$ if and only if $x\theta y\theta z$ or $z\theta y\theta x$ (where $x\theta y$ denotes $(x,y) \in \theta$) is an order-betweenness relation in X, that becomes total if θ is a total order. B_θ is called the order-betweenness relation associated to θ .

We can find a constructive method in [3] to obtain an order relation from an order-betweenness relation B (unique unless duality in each connected class).

J. Gilder's axioms ([2]) for the ternary betweenness relation are the following:

(G.1) $(a,b,c)\in B$ and $(a,c,b) B \Leftrightarrow b=c$,

(G.2) $(a,b,c)\in B \Rightarrow (c,b,a)\in B$,

(G.3) $(a,b,c)\in B$ and $(a,c,d)\in B \Rightarrow (b,c,d)\in B$.

Axioms (PB.1) and (PB.2) imply (G.1), (G.2) and (G.3), but the reciproc does not hold. For example, the classical metric betweenness relation ([4]) defined in a metric space (X,d) "$(x,y,z)\in B_d \Leftrightarrow d(x,z) = d(x,y)+d(y,z)$" satisfies (G.1), (G.2) and (G.3), but B_d does not verify (PB.2).

One may wonder whether, by adding property (T.A.) in both axiomatic systems, these become equivalent. This is not the case, since a "cyclic-betweenness" ([2]) satisfies the four Gilder axioms, but does not verify (PB.2).

On binary relations infered from a betweenness relation, by fixing one element of the triplets, we can enunciate the following

<u>Theorem:</u> Let B be a partial-betweenness relation in a non-empty set X.

(i) Given any $b\in X$ the binary relation B^b defined by $x B^b y \Leftrightarrow (x,b,y)\in B$ or $x=y$" is a reflexive and symmetric relation. If $B = B_\theta$ is the partial betweenness relation associated to a partial order relation θ and b is a extremal <u>element</u>, then B^b is the comparability relation, and its transitive closure $\overline{B^b}$ is the connectedness relation.

(ii) Given any $a\in X$, the binary relation B_a defined by "$x B_a y \Leftrightarrow (a,x,y)\in B$ or $x=y$", is a partial order relation in X and a is a minimal element.

If θ is a partial order relation in X and B_θ its associated partial-betweenness relation:

(iii) $(B_\theta)_a \subset \theta$ (resp. $(B_\theta)_a \subseteq \theta^t$, where θ^t, denotes the dual relation of θ), if and only if a is a minimal (resp. maximal) element in the partially ordered set (X,θ).

(iv) If the ordered set (X,θ) is connected, then $(B_\theta)_a = \theta$ (resp. $(B_\theta)_a = \theta^t$), if and only if a is the minimum (resp. the maximum) of (X,θ).

We shall remark that, in general, if B is a total-betweenness relation, the binary relation B_a is not total. For example, in $(N, \leqslant)$ the relation $B = B_{\leqslant}$ is total, while $1 \in N$ is not comparable with the other natural numbers by the binary relation B_2.

Note that the connectedness of (X,θ) plays a crucial role in (iv) because the incomparable elements with a for θ (resp. for θ^t) are also incomparable for $(B_\theta)_a$, even when $(B_\theta)_a=\theta$ (resp. $(B_\theta)_a=\theta^t$), and, thus, a cannot be the minimum (resp. maximum).

Clifford ([1]) and Gilder ([2]) definitions on partially ordered semigroups of the first, second and third kinds, can be generalized without dificulties for monoids, and the same can be done for the concept of betweenness semigroup. M. Jalobeanu ([3]) proved that all order betweenness semigroup, wich is strict and connected, is necessarily a partially ordered semigroup of the first, second or third kind. This result is generalizable for monoids.

Theorem: (i) If $(X,\cdot,B)$ is a strong partial betweenness monoid ([3]) such that all elements are cancellable, then $(X,\cdot,B)$ is a strict betweenness monoid ([3]). (ii) If $(X,\cdot,B)$ is a partial-betweenness semigroup with neutral such that cancellable elements agree with inversible elements, then $(X,\cdot,B)$ is a strong partial-betweenness semigroup; in particular, if $B=B_\theta$ is given by a partial order, $(X,\cdot,B_\theta)$ is a strong order-betweenness semigroup whenever $(X,\cdot,\theta)$ is a connected partially ordered semigroup.

As a corollary it is obvious that if $(X,\cdot,\theta)$ is a partially ordered group, there exist equivalence between the concepts of strong, strict and partially ordered of first, second or third kinds.

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