STOCHASTICA, Vol.III nº 1 (1979)

ON SOME TOPOLOGIES ON A GENERALIZED METRIC SPACE

by

Montserrat Pons

Introduction.

If $(\Omega,\mathcal{P},m)$ is a generalized metric space (g.m.s.), we define, for a certain class of subsets S of $\mathcal{P}$, the $T_m(S)$ topology on Ω and study its properties. We point out that, for a product of g.m.s. $(\Omega_i,\mathcal{P}_i,m_i)_{i\in I}$, a $T_m(S)$ topology on $\prod_{i\in I}\Omega_i$ is obtained from $T_{m_i}(S_i)$ topologies on each factor.

$T_m(S)$ topologies on an abelian l-group G are considered. We show that the open-interval topologies arising from compatible tight Riesz orders on G are of this type.

1. The $T_m(S)$ topology.

Let (Ω,φ,m) be a generalized metric space [5], such that $\mathcal{P}=(\mathcal{P},+,\leqslant)$ is an abelian partially ordered semigroup where the zero element is the minimum. We shall deal with non empty subsets S of $\mathcal{P}$ not containing the zero element.

A subset $A\subset\Omega$ is said to be $T_m(S)$-open if, for every $a\in A$, there exists $r\in S$ such that $B_a(r)\subset A$. The $T_m(S)$-open sets form a collection, containing $\emptyset$ and Ω, closed by arbitrary joins. In general, it is not closed by finite meets. A sufficient condition for the $T_m(S)$-open sets to be a topology on Ω is that S be directed downwards; nevertheless, this condition does not ensures that the balls with radius in S be a local base of neighbourhoods at each point. This will be the case when S also satisfies the following pseudo-divisibility condition: for every

$r\in S$, there exists $s\in S$ such that $s+s<r$ (refered as pseudo-radicals condition in [1]). All this can be stated in:

1.1. <u>Proposition:</u> Let S be directed downwards and pseudo-divisible. Then, the collection of $T_m(S)$-open sets is a topology. Moreover, the balls with radius in S and centered at $a\in\Omega$ form a local base at a. This topology $T_m(S)$ is uniformizable and, if m is separating and inf S=0, it is Hausdorff.

Proof: It is obvious that $a\in B_a(r)$ $\forall a\in\Omega$, $\forall r\in S$, and that $t\leqslant s,r$ implies $B_a(r)\cap B_a(s)\supset B_a(t)$. Given $B_a(r)$, with $r\in S$, we can find $s\in S$ such that $s+s<r$, and, if $y\in B_a(S)$, then $B_y(s)\subset B_a(r)$. Thus, the r-balls ($r\in S$) form a local base for a topology which is the $T_m(S)$ topology.

Furthermore, the collection of sets $\{D_r\}_{r\in S}$, where $D_r = \{(a,b)\in\Omega x\Omega \mid m(a,b)<r\}$, is a base for an uniformity $\mathcal{D}$ on Ω whose generated topology is $T_m(S)$.

If m is separating and inf S=0, then $\bigcap_{r\in S} D_r=\Delta$. Hence $\mathcal{D}$ is separating and $T_m(S)$ is Hausdorff.

1.2. <u>Corollary:</u> Let S be directed downwards, pseudo-divisible and containing a countable dense subset M (in the sense that each element in S exceeds someone in M). Then, the $T_m(S)$ topology is pseudo-metrizable and, if m is separating and inf S=0, it is metrizable.

Proof: $T_m(S)$ is uniformizable, by 1.1, and the collection of sets $\{D_r\}_{r\in M}$ forms a countable base for $\mathcal{D}$, so $T_m(S)$ is pseudo-metrizable. By adding the other conditions $T_m(S)$ is Hausdorff and hence metrizable.

2. $T_m(S)$ topologies on the product of generalized metric spaces.

Let $(\Omega,\mathcal{P},m)$ be the product of a collection $(\Omega_i,\mathcal{P}_i,m_i)_{i\in I}$ of generalized metric spaces [5].

2.1. Proposition: If $S_i \subset \mathcal{S}_i$ is directed downwards and pseudo-divisible for each $i \in I$, then $S = \prod_{i \in I} S_i \subset \mathcal{S}$ also satisfies these properties, and the $T_m(S)$ topology on Ω is the box topology obtained from the $T_{m_i}(S_i)$ topologies on each factor.

Proof: It is straighforward to prove S to be directed downwards and pseudo-divisible.

Let T be the box topology on Ω obtained from the $T_{m_i}(S_i)$ topologies on each factor. If $A \subset \Omega$ is a $T_m(S)$-open and $a = (a_i)_{i \in I} \in A$, there exists $s = (s_i)_{i \in I} \in S$ such that $B_a(s) \subset A$. For each $i \in I$, $B_{a_i}(s_i)$ is a $T_{m_i}(S_i)$-nhood of a_i, and, hence, there exists a $T_{m_i}(S_i)$-open G_i such that $a_i \in G_i \subset B_{a_i}(s_i)$. Thus, $a \in \prod_{i \in I} G_i \subset A$ and $\prod_{i \in I} G_i$ is a T-open.

Conversely, let A_i be a $T_{m_i}(S_i)$-open for each $i \in I$, and consider $A = \prod_{i \in I} A_i$. If $a = (a_i)_{i \in I} \in A$, there are some elements $s_i \in S_i$ $\forall i \in I$, such that $B_{a_i}(s_i) \subset A_i$. For each s_i we can find $r_i < s_i$ $(r_i \in S_i)$, by the pseudo-divisibility, and, if $r = (r_i)_{i \in I}$, then $B_a(r) \subset A$ and $r \in S$. Therefore, $T_m(S) = T$.

3. $T_m(S)$ topologies on abelian l-groups.

Let $(G, +, \leq)$ be an abelian l-group and (G, G^+, m) its associated generalized metric space.

3.1. Proposition: If S is directed downwards and pseudo-divisible, then G is a topological l-group under the $T_m(S)$ topology which is not discrete. It is Hausdorff if and only if $\inf S = 0$.

Proof: Since $|x| = |-x|$ $\forall x \in G$, the neighbourhoods at zero are symmetric. The pseudo-divisibility ensures that, for each $s \in S$, there exists $r \in S$ such that $r < r+r < s$, hence $B_0(r) + B_0(r) \subset B_0(s)$, and, since $a + B_0(r) = B_a(r)$, G is a topological group. Furthermore, $B_x(r) \vee B_y(r) \subset B_{x \vee y}(s)$ and, consequently, the map $(a,b) \to a \vee b$ is continuous. Therefore, G being a topological group, G is also a topological lattice under the

$T_m(S)$ topology, which is not discrete due to pseudo-divisibility.

Finally, (prop. 1.1) inf S=0 implies $T_m(S)$ is Hausdorff. Conversely, if r is an element such that $0 < r < s$ $\forall s \in S$, we have $r \in B_o(s)$ $\forall s \in S$ and, then, the $T_m(S)$ topology is not Hausdorff.

These topologies are deeply related to the open-interval topologies on an l-group obtained from the compatible tight Riesz orders we can define on it. Let us recall the following definitions and basic properties [2][6]:

A partially ordered group $(G,\leqslant)$ is said to be a tight Riesz group (TRG) when it is directed and has the tight Riesz property: for every four elements $a_1 a_2 b_1 b_2$ in G, such that $a_i < b_j$ for i,j=1,2, there exists $c \in G$ such that $a_i < c < b_j$ for i,j=1,2.

A tight Riesz group is a topological group under its open-interval topology $\mathcal{U}_{\leqslant}$ which is not discrete. If the closure of its positive cone is the positive cone of an ordering $\preccurlyeq$, we say $\preccurlyeq$ is the associated order.

Let $(G,\preccurlyeq)$ be an l-group. A compatible tight Riesz order (CTRO) on $(G,\preccurlyeq)$ is a partial ordering $\leqslant$ making $(G,\leqslant)$ a tight Riesz group and having $\preccurlyeq$ as its associated order.

3.2. <u>Proposition:</u> Let $(G,\preccurlyeq)$ be an abelian 1-group and $S \subset G^+ | \{0\}$ be: i) directed, ii) pseudo-divisible and such that iii) inf S=0. Then, the $T_m(S)$ topology is the open-interval topology associated with some CTRO on $(G,\preccurlyeq)$. Conversely, given a CTRO on $(G,\preccurlyeq)$, there exists S with the above properties such that $T_m(S)$ is its associated open-interval topology.

Proof: If S satisfies the former conditions, then

$$S^* = \{x \in G \mid \exists s \in S \text{ such that } x \succcurlyeq s\}$$

is the strictly positive cone of a CTRO $\leqslant$ on $(G,\preccurlyeq)$, since we can easily verify that the "Wirth's conditions" [6] hold.

Now, if $a \in S^*$, there exists $s \in S$ such that $a \succcurlyeq s$, and $B_o(s) \subset \{x \in G | -a < x < a\}$. Besides, if $s \in S$, then $s \in S^*$ and $\{x \in G | -s < x < s\} \subset B_o(s)$. Therefore, $T_m(S) = \mathcal{U}_{\leqslant}$.

Conversely, if P is the strictly positive cone for a CTRO $\leqslant$ on $(G,\preccurlyeq)$, then P obviously satisfies the i) ii) and iii) conditions. Consequently, $T_m(P) = \mathcal{U}_{\leqslant}$ as P coincides with P*.

REFERENCES

[1] BATLE, N., "Contribución a un estudio básico de los espacios métricos probabilísticos". Tesis doctoral. Barcelona, 1973.

[2] LOY, R. J., MILLER, J. B., "Tight Riesz groups". J. Australian Math. Soc. 13 (1972), 224-240.

[3] PONS, M., "Un estudi sobre topologies en espais mètrics de Riesz". Tesina. Barcelona, 1976.

[4] SMARDA, B., "Topologies in 1-groups". Arch. Math. (Brno) 3 (1967), 69-81.

[5] TRILLAS, E., ALSINA, C., "Introducción a los espacios métricos generalizados". Publ. Funf. Juan March, Serie universitaria 49 (1978).

[6] WIRTH, A., "Compatible tight Riesz orders". J. Australian Math. Soc., 15 (1973), 105-111.

Departament de Matemàtiques i Estadistica
E.T.S. d'Arquitectura
Universitat Politècnica de Barcelona