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A FORMALIZATION OF THE LEWIS SYSTEM S1 WITHOUT RULES OF SUBSTITUTION

by

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0. Summary

In the Lewis and Langford formalization of system S1 (1932) are used besides the deduction rules, the substitution rules: the uniform substitution and the substitution of strict equivalents. They obtain then systems S2, S3, S4 and S5 adding to the axioms of S1 a new axiom, respectively, without changing the deduction rules. Lemmon (1957) gives a new formalization of systems S1-S5, calling P1-P5. It is worthwhile to remark that in the formalization of P2-P5 one does not use any more the substitution of equivalents rule, although Lemmon still maintains the uniform substitution rule. Anyhow Lemmon system P1 uses the substitution of equivalents rule in addition to uniform substitution rule. Moreover these substitution rules have been used later by Feys (1965), Hughes and Cresswell (1968), Zeman (1973) to construct Lewis modal systems.

This paper deals with a new formalization of S1 system, following Lemmon ideas, without substitution rules.

1. Lewis system S1 without uniform substitution.

Let M(X) be the free algebra in the class A of algebras of type (1,1,2) generated be X. Operations in M(X) are denoted by $\neg, \llcorner$ (monadic) and $\wedge$ (binary). As usual, one defines in M(X)

$$
\begin{aligned}
p \rightarrow q &= \neg(p\wedge\neg q),\\
p \leftrightarrow q &= (p \rightarrow q) \wedge (q \rightarrow p),\\
p \vee q &= \neg(\neg p\wedge\neg q),\\
p \Rightarrow q &= L(p \rightarrow q),\\
p \Leftrightarrow q &= (p \Rightarrow q) \wedge (q \Rightarrow p),\\
Mp &= \neg L\neg p.
\end{aligned} \tag{1}
$$

Let us call PC (classical propositional calculus), all thesis obtained from axioms

$$
\begin{aligned}
&a_1\colon\ x_1 \wedge x_2 \rightarrow x_1,\\
&a_2\colon\ x_1 \rightarrow x_1 \wedge x_1,\\
&a_3\colon\ x_1 \wedge x_2 \rightarrow x_2 \wedge x_1,\\
&a_4\colon\ (x_1 \rightarrow x_2) \rightarrow (\neg(x_2 \wedge x_3) \rightarrow \neg(x_1 \wedge x_3))
\end{aligned} \tag{2}
$$

where $x_i \in X$, $i=1,2,3$, and uniform substitution rule and modus ponens (cf. Hughes and Cresswell, op.cit.).

Let us call PC* (classical propositional calculus) all thesis obtained from axioms

$$
\begin{aligned}
&A_1=\{p\wedge q \rightarrow p : p,q \in M(X)\},\\
&A_2=\{p \rightarrow p\ p : p \in M(X)\},\\
&A_3=\{p\wedge q \rightarrow q\wedge p : p,q \in M(X)\},\\
&A_4=\{(p \rightarrow q) \rightarrow (\neg(q\wedge r) \rightarrow \neg(p\wedge r)) : p,q,r \in M(X)\}
\end{aligned} \tag{3}
$$

and modus ponens (m.p.).

By induction on the length of the proof one shows that

$$PC = PC^*. \tag{4}$$

We denote P1 the class of all thesis of Lemmon system P1:

> $P \in P1$ if, and only if, p is obtained from p_1 or p_2 using the uniform substitution rule, modus ponens, rule of weak necessity (N.S1) and substitution of strict equivalents rule (cf. Hughes and Cresswell, ep.cit.) where p_1 is $((x_1 \Rightarrow x_2)\wedge(x_2 \Rightarrow x_3)) \rightarrow (x_1 \Rightarrow x_3)$ and p_2 is $Lx_1 \Rightarrow x_1$, where x_1, x_2, $x_3 \in X$.

Let Q1 be the class of thesis of the abstract logic $(M(X),\Gamma)$ where Γ is the class of sets $D \subseteq M(X)$ such that

d_1. $B_1 \cup B_2 \subseteq D$,

d_2. if $p \in PC^*$ or $p \in B_1 \cup B_2$, then $Lp \in D$ (N.Q1),

d_3. if $p \in D$ and $p \rightarrow q \in D$, then $q \in D$ (m.p.),

d_4. if p differs from q only in having a formula r in some of the places where q has s, then if $(r \Leftrightarrow s) \in D$, then $(p \Leftrightarrow q) \in D$, (S.S.),

where $B_1=\{((p \Rightarrow q)\wedge(q \Rightarrow r)) \rightarrow (p \Rightarrow r): p,q,r \in M(X)\}$ and $B_2=\{Lp \rightarrow p: p \in M(X)\}$.

<u>Theorem.</u> P1=Q1.

The proof proceeds by induction on the length of the formal proof, and has no difficulty because of the equality PC=PC*. From now on we shall use the characterization Q1 of P1 and we shall call N.Q1 the weak necessity rule.

2. The Lewis system S1 without substitution rules.

As usual, let us call sub (p) the smallest subset A of M(X) such that

s_1. $p \in A$,

s_2. if $\neg q \in A$, then $q \in A$,

s_3. if $L q \in A$, then $q \in A$,

s_4. if $q_1 \wedge q_2 \in A$, then $q_1 \in A$ and $q_2 \in A$.

The set sub (p) is called the set of subformulas of p and every formula $q \in$ sub (p) is called a subformula of p.

If $p \in M(X)$ and $q, s \in M(X)$ we define

$$[p]_q^s$$

by

$$[x]^s_q = \begin{cases} \{x\} & \text{if } q \neq x \\ \{x,s\} & \text{if } q = x, \end{cases}$$

$$[p]^s_q = \begin{cases} \{\neg r\}: r\in[p]^s_q & \text{if } q\in \text{sub }(p) \\ \{\neg p,\ s\} & \text{if } q = \neg p \\ \{\neg p\} & \text{if } q \notin \text{sub }(\neg p), \end{cases}$$

$$[Lp]^s_q = \begin{cases} \{Lr: r\in[p]^s_q & \text{if } q\in \text{sub }(p) \\ \{Lp,\ s\} & \text{if } q = Lp \\ \{Lp\} & \text{if } q \notin \text{sub }(Lp), \end{cases}$$

$$[p\wedge p']^s_q = \begin{cases} \{r\wedge r': r\in[p]^s_q \text{ and } r'\in[p']^s_q & \text{if } q\in \text{sub}(p\wedge p')-\{p\wedge p'\} \\ \{p\wedge p', s\} & \text{if } q = p\wedge p' \\ \{p\wedge p'\} & \text{if } q\in \text{sub }(p\wedge p'). \end{cases}$$

If $r\in[p]^s_q$, we say that r is obtained form p by substitution of a collection of instances (possibily empty) of q by s.

It is easy to see that, if $r\in[p]^s_q$ and $r'\in[p']^s_q$, then

$$Lr\in[Lp]^s_q,\ r \Rightarrow r'\in[p \Rightarrow p']^s_q, \text{ etc.} \quad (5)$$

We can reformulate rule S.S.:

$$\text{if } r \Leftrightarrow s\in D \text{ and } p'\in[p]^s_r, \text{ then } p \Leftrightarrow p'\in D. \quad (6)$$

Let us call

c_1. if $p \Leftrightarrow q\in D$, then $\neg p \Leftrightarrow \neg q\in D$,

c_2. if $p \Leftrightarrow q\in D$, then $p\wedge r \Leftrightarrow q\wedge r\in D$, (7)

c_3. if $p \Leftrightarrow q\in D$, then $Lp \Leftrightarrow Lq\in D$ (rule C.)

These rules say that the equivalence relation

$$p \sim q \quad \text{if, and only if,} \quad p \Leftrightarrow q\in D,\quad D\in\Gamma$$

is compatible with the operations of M(X); that is, the quotient M(X)/D is an algebra.

It is easy to show by induction on the length of the formula that, for all subset $D \subseteq M(X)$,

d_1, d_2, d_3 and c_1, c_2, c_3 if, and only if, d_1, d_2, d_3, d_4.

The proof is rather cumbersome.

Moreover, one can prove:

<u>Theorem.</u> For all subset $D \subseteq M(X)$,

d_1, d_2, d_3 and rule C if, and only if, d_1, d_2, d_3, d_4.

Proof. Part if is trivial. We proceed with the only if.

i. $PC^* \subseteq D$.
If $p \in PC^*$, then $Lp \in D$ (by d_2); but $Lp \rightarrow p \in D$ (by d_1) and (by d_3) $p \in D$.

ii. If $p \in D$ and $q \in D$, then $p \wedge q \in D$. (Adjunction rule.)
By virtue of (i)

$$p \rightarrow (q \rightarrow (p \wedge q)) \in PC^* \subseteq D.$$

From d_3 we obtain: $p \wedge q \in D$.

iii. If $p \leftrightarrow q \in PC^*$, then $p \Leftrightarrow q \in D$.
If $p \leftrightarrow q \in PC^*$, then $p \rightarrow q \in PC^*$ and $q \rightarrow p \in PC^*$. By d_2, $p \Rightarrow q \in D$ and $q \Rightarrow p \in D$, as claimed (by (ii)).

iv. $(\neg p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow p) \in D$.
We have

$$(\neg p \rightarrow q) \Leftrightarrow (\neg q \rightarrow p) \in D \text{ (by (iii))},$$

and we can apply <u>rule C.</u>

v. $p \Leftrightarrow \neg\neg p \in D$.
See proof of iv.

From (iv) and (v) we obtain:

iv'. $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p) \in D$.

vi. If $p \in D$ and $p \Rightarrow q \in D$, then $q \in D$ (modus ponens for strict implication).
If $p \Rightarrow q \in D$, then (by (i), d_1 and d_3) $p \rightarrow q \in D$.
Thus (by d_3): $q \in D$.

vii. If $p \Rightarrow q \in D$ and $q \Rightarrow r \in D$, then $p \Rightarrow r \in D$ (transitivity rule).
If $p \Rightarrow q \in D$ and $q \Rightarrow r \in D$ we obtain (by (ii)) $(p \Rightarrow q) \wedge (q \Rightarrow r) \in D$.
Thus (by d_1 and (iv)): $p \Rightarrow r \in D$.

viii. $p \wedge q \Rightarrow p \in D$.
Obvious.

ix. If $p \Leftrightarrow q \in D$, then $p \Rightarrow q \in D$.
Obvious from (viii).

Now we obtain c_1: if $p \Leftrightarrow q \in D$, then we have $\neg\neg p \Leftrightarrow q \in D$ (by (v) and t.r.). We apply (iv), (ix) and (vi): $\neg q \Leftrightarrow \neg p \in D$. By simetry, we have finally:

$$\neg p \Leftrightarrow \neg q \in D.$$

Actually, we have seen more: if $p \Rightarrow q \in D$, then $\neg q \Rightarrow \neg p \in D$. (8)

x. $p \Leftrightarrow p \in D$, $p \wedge q \Rightarrow p \in D$, $p \Rightarrow p \wedge p \in D$ and $(p \wedge (q \wedge r)) \Leftrightarrow ((p \wedge q) \wedge r) \in D$.
Obvious.

xi. $p \Leftrightarrow (\neg p \rightarrow p) \in D$.
Obvious.

xii. $Lp \Leftrightarrow (\neg p \Rightarrow p) \in D$.
By rule C applied to (xi).

xiiii. $((p \wedge q) \Rightarrow r) \Leftrightarrow (p \Rightarrow (q \rightarrow r)) \in D$.
$((p \wedge q) \rightarrow r) \Leftrightarrow (p \rightarrow (q \rightarrow r)) \in PC^*$. Thus, by N.Q1 and rule C we conclude the assertion.

xiv. $((p \Rightarrow q) \wedge (q \Rightarrow r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow s) \in D$.
We have $((p \Rightarrow r) \wedge (r \Rightarrow s)) \Rightarrow (p \Rightarrow s) \in D$ (by d_1 and d_2).
By (xiii) we obtain: $(p \Rightarrow r) \Rightarrow ((r \Rightarrow s) \rightarrow (p \Rightarrow s))$ D.
But (by d_1 and d_2) we have: $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r) \in D$.
And (by t.r) we conclude: $((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow ((r \Rightarrow s) \rightarrow (p \Rightarrow s)) \in D$.
And we can apply (xiii) and modus ponens for estrict implication.

xv. $L(p \rightarrow q) \rightarrow (Lp \rightarrow Lq) \in D$ and $L(p \rightarrow q) \Rightarrow (Lp \rightarrow Lq) \in D$.
By (xiv) : $((\neg q \Rightarrow \neg p) \wedge (\neg p \Rightarrow p) \wedge (p \Rightarrow q)) \Rightarrow (\neg q \Rightarrow q) \in D$.
By (xii) and (t.r): $((\neg q \Rightarrow \neg p) \wedge (\neg p \Rightarrow \neg p) \wedge (p \Rightarrow q)) \Rightarrow Lq \in D$.
By (xiii): $((\neg q \Rightarrow \neg p) \wedge (\neg p \Rightarrow p)) \Rightarrow ((p \Rightarrow q) \rightarrow Lq) \in D$.
By (x) : $((\neg p \Rightarrow p) \wedge (\neg q \Rightarrow \neg p)) \Rightarrow ((p \Rightarrow q) \rightarrow Lq) \in D$.
By (xiii): $(\neg p \Rightarrow p) \Rightarrow ((\neg q \Rightarrow \neg p) \rightarrow ((p \Rightarrow q) \rightarrow Lq)) \in D$.
By (xii) : $Lp \Rightarrow ((\neg q \Rightarrow \neg p) \rightarrow ((p \Rightarrow q) \rightarrow Lq) \in D$.
By (xiii): $((Lq \wedge (\neg q \Rightarrow \neg p)) \Rightarrow ((p \Rightarrow q) \rightarrow Lq) \in D$.
By (x) : $((\neg q \Rightarrow \neg p) \wedge Lp) \Rightarrow ((p \Rightarrow q) \rightarrow Lq) \in D$.
By (xiii): $(\neg q \Rightarrow \neg p) \Rightarrow (Lp \rightarrow ((p \Rightarrow q) \rightarrow Lq)) \in D$.
By (iv') : $(p \Rightarrow q) \Rightarrow (Lp \rightarrow ((p \Rightarrow q) \rightarrow Lq))$ D.
By (xiii): $((p \Rightarrow q) \wedge Lp \wedge (p \Rightarrow q)) \Rightarrow Lq \in D$.

By (x) : $((p \Rightarrow q)\wedge(p \Rightarrow q)\wedge Lp) \Rightarrow Lq \in D$.

By (x) : $((p \Rightarrow q)\wedge(p \Rightarrow q)) \Leftrightarrow (p \Rightarrow q) \in D$.

By (8) : $((p \Rightarrow q)\wedge(p \Rightarrow q)\wedge Lp) \Leftrightarrow ((p \Rightarrow q)\wedge Lp) \in D$.

By t.r. : $((p \Rightarrow q)\wedge Lp) \Rightarrow Lq \in D$.

By (xiii) : $(p \Rightarrow q) \Rightarrow (Lp \rightarrow Lq) \in D$.

Eventually, we obtain the following rule:

$$\text{If } p \Rightarrow q \in D, \text{ then } Lp \rightarrow Lq \in D. \quad (9)$$

xvi. $(p \rightarrow q) \Rightarrow (\neg(q\wedge r) \rightarrow \neg(p\wedge r)) \in D$.

$(p \rightarrow q) \rightarrow (\neg(q\wedge r) \rightarrow \neg(p\wedge r)) \in RC^*$. By N.Q1 we obtain the proposed formula.

Now we apply the rule (9) to (xvi) and we obtain:

$$(p \Rightarrow q) \rightarrow (\neg(q\wedge r) \Rightarrow \neg(p\wedge r)) \in D.$$

<u>Finally, we have c_2</u>:

as $p \Rightarrow q \in D$ implies $\neg(q\wedge r) \Rightarrow \neg(p\wedge r) \in D$

we obtain (by (iv')): $(p\wedge r) \Rightarrow (q\wedge r) \in D$.

This ends the proof.

3. References.

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