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A MARKOV PROPERTY FOR TWO PARAMETER GAUSSIAN PROCESSES (*)

by

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This paper deals with the relationship between two-dimensional parameter Gaussian random fields verifying a particular Markov property and the solutions of stochastic differential equations. In the non Gaussian case some diffusion conditions are introduced, obtaining a backward equation for the evolution of transition probability functions.

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First we examine the Gaussian case and we establish a characterization of random Markov fields by means of two-parameter Wiener processes.

Let $X=\{X(z)\}_{z\in R^2}$ be a random field, defined on a probability space $(\Omega, \mathcal{A}, P)$. If the random field is only defined on a subset $T\subset R^2$ we will extend it by putting $X(z)=0$ if $z\notin T$.

We consider two increasing families of σ-fields:

$$F_z = \sigma < X(z'), z' \leqslant z > ,$$

$$F_z = \sigma < X(z'), x' \leqslant x \quad \text{or} \quad y' \leqslant y > , \quad z=(x,y), \quad z'=(x',y'),$$

which have been defined using the partial ordering of R^2:

$$(x_1, y_1) \leqslant (x_2, y_2) \Leftrightarrow x_1 \leqslant x_2 \quad \text{and} \quad y_1 \leqslant y_2 .$$

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Given a point $z \in R^2$, the complementary regions $\{z' \in R^2 \mid z \leq z'\}$ and $\{z' \in R^2 \mid x' < x \text{ or } y' < y\}$ represent respectively the "future" and "past" of z.

Definition 1.1.- The random function X is called a two-parameter Markov random field iff

$$P[X(z')/G_z] = P[X(z')/X(x,y'), X(z), X(x',y)],$$

for all $z \leq z'$.

Notice that the two-parameter Wiener process $W=\{W(z)\}_{z \in R^2_+}$ and any random field on R^2_+ with independent increments, zero mean and vanishing on the axes satisfy this property.

Let $F^1_{st} = \bigvee_{y \geq 0} F_{sy}$ and $F^2_{st} = \bigvee_{x \geq 0} F_{xt}$ for all $(s,t) \in R^2$. We will assume that the family of σ-fields $\{F_z\}_{z \in R^2_+}$ associated to a random field $X = \{X(z)\}_{z \in R^2_+}$ satisfies the following conditions:

(1.1) F^1_{st} and F^2_{st} are conditionally independent given F_z.

(1.2) $\bigwedge_{s \in R} F^1_{st}$ and $\bigwedge_{t \in R} F^2_{st}$ are trivial σ-fields.

It can be shown (see [3] for the Gaussian case) that Markov property implies (1.1), and (1.2) is true if, for example, the process varishes in a region $Q_{z_o} = \{(x,y) \mid x \leq x_o \text{ or } y \leq y_o\}$ for some $z_o \in R^2$.

We suppose in this section that X is a Gaussian, zero mean random field and we will denote by $\Gamma(z,z')$ its covariance function. Let $\Gamma(z)$ be the variance of X(z) and let R_z be the rectangle $(-\infty,x] \times (-\infty,y]$ for all $z=(x,y) \in R^2$.

We first give a characterization of Gaussian martingales.

Proposition 1.1.- The following conditions are equivalent:

(i) $\{X(z), F_z\}_{z \in R^2}$ is a martingale,

(ii) X has independent increments,

(iii) $\Gamma(z,z') = \Gamma(z \wedge z')$, where $z \wedge z' = (x \wedge x', y \wedge y')$.

If $\Gamma(z)$ is absolutely continuous with respect to the Lebesgue measure on R^2 then the previous conditions are also equivalent to:

(iv) there exists a Brownian measure W on R^2 and a function $\phi\in L^2(R^2)$ such that

(1.3) $X(z) = \int_{R_z} \phi(\zeta)\, dW_\zeta$.

Proof: We will only demonstrate the three first conditions imply the fourth one, because other implications are obvious.

If $\Gamma(z)=\int_{R_z}\varphi(\zeta)d\zeta$, property (iii) implies the increment of Γ in any rectangle Δ es nonnegative, that is, $\Gamma(\Delta)=E[X(\Delta)]^2 \geq 0$, therefore we can suppose $\varphi(\zeta)\geq 0$ for all $\zeta\in R^2$. and take $\phi(\zeta)=\sqrt{\varphi(\zeta)}$.

Let us consider two cases:

(a) If $\phi(\zeta)\neq 0$ for all $\zeta\in R^2$, then $W(A)=\int_A \frac{1}{\phi(\zeta)}\, dX(\zeta)$, where A is a bounded measurable subset of R^2, defines a Brownian measure on R^2 which satisfies (1.3).

(b) If ϕ varishes in some point ζ , we take $\frac{1}{\phi(\zeta)} = 0$.

Let W_o be a Brownian measure on R^2 independent of X (it can always be consructed modifying the probability space $(\Omega,\mathcal{a},P)$) and D the set $\{\zeta\in R^2/\phi(\zeta)=0\}$. Then property (iv) follows from

$$W(A)=\int_A \frac{1}{\phi(\zeta)}\, dX(\zeta) + W_o(D\cap A). \quad \square$$

Notice that property (iv) imply conditions (1.1) and (1.2).

Let $\phi(z,z')$ a real function defined by

$$\phi(z,z')=\Gamma(z,z')\Gamma(z',z')^{-1} \text{ if } z'\leq z,\ z'\neq z \text{ and } \Gamma(z',z')\neq 0,$$
$$\phi(z,z')=0 \qquad \text{if } z'\leq z,\ z'\neq z \text{ and } \Gamma(z',z')=0,$$
$$\phi(z,z) =1.$$

ϕ represents a transmission function, as in the one dimensional parameter processes, and satisfies

$$E\,[X(z)/X(z')] = \phi(z,z').X(z'), \text{ if } z'\leq z.$$

Proposition 1.2.- The process $X = \{X(z)\}_{z \in R^2}$ is a Markov random field iff

(1.4) $\Gamma(z,z") = \phi(z,(x',y))\ \Gamma((x',y),z")$, for all $x" \leqslant x' \leqslant x$, and

(1.5) $\Gamma(z,z") = \phi(z,(x,y'))\ \Gamma((x,y'),z")$, for all $y" \leqslant y' \leqslant y$,

where $z=(x,y)$, $z"=(x",y")$.

In this case the function ϕ verifies

(1.6) $\phi(z,z") = \phi(z,z')\phi(z',z")$ if $z" \leqslant z' \leqslant z$.

As a consequence of this proposition we obtain that under condition (1.2) definition 1.1. is equivalent, in the Gaussian case, to the Markov property introduced by H. Korezlioglu in [3].

Proof: The Markov property, in the Gaussian case, is equivalent to the existence, for all $z' \leqslant z$, of real numbers $\lambda_1, \lambda_2, \lambda_3$ such that

$$E[X(z)/G_{z'}] = \lambda_1 X(x,y') + \lambda_2 X(z') + \lambda_3 X(x',y);$$

that means

(1.7) $\Gamma(z,z") = \lambda_1 \Gamma((x,y'),z") + \lambda_2 \Gamma(z',z") + \lambda_3 \Gamma((x',y),z")$

for all $z' \in Q_z$.

Then, it is immediate that conditions (1.4) and (1.5) imply (1.7) and the reciproque follows taking $x' \to -\infty$ or $y' \to -\infty$ in (1.7) and applying property (1.2). □

In order to obtain a characterization of X in the Markov case, we need several smoothing hypotheses about its covariance function.

H1.- $\Gamma(z,z')$ is a continuous function and the set $D=\{z \in R^2 / \Gamma(z) > 0\}$ is either R^2 or $Q_{z_0}^c = \{z \in R^2 / z >> z_0\}$ for a certain point z_0, where $z_0 << z$ means $x_0 < x$ and $y_0 < y$.

H2.- There exists a positive continuous function $f: \bar{D} \to R$ such that $\phi(z,z') = f(z), f(z')^{-1}$ for all $z' \leqslant z$, $z', z \in D$.

H3.- $K(z) = f(z)^{-2} . \Gamma(z)$ is an absolutely continuous function on D.

Theorem 1.1.- If the covariance function $\Gamma(z,z')$ of the process X verifies the previous hypotheses H1, H2, H3, X is a Markov random field iff there exist a Brownian measure W on R^2, a continuous function $\phi_1(z)$ on R^2 and a square integrable function $\phi_2(z)\in L^2(R^2)$ such that

(1.8) $X(z)=\phi_1(z).\int_{R_z}\phi_2(\zeta)dW(\zeta)$.

Proof: The random function $Y(z)=f(z)^{-1}.X(z)$ if $z\in D$, $Y(z)=0$ if $z\notin D$, has a covariance function

$$E[Y(z).Y(z')] = f(z\wedge z')^{-2}.\Gamma(z\wedge z'),$$

which satisfies (iii) of Proposition 1.1, because of hypothesis H3.

Therefore, there exist a Brownian measure W and a function $\phi_2(z)\in L^2(R^2)$ such that $Y(z)=\int_{R_z}\phi_2(\zeta)dW(\zeta)$. It suffices, then, to take $\phi_1(z)=f(z)$.

Conversely, it is clear that every random field X(z) given by (1.8) satisfies the Markov property. □

Suppose that for all $z=(x,y)$, $z'=(x',y')$, $z'\leqslant z$, $z,z'\in D$ there exist the following partial derivatives

$$b(z)=-\frac{\partial\phi(z,z')}{\partial x^+}\Big|_{z=z'},\quad a(z)=-\frac{\partial\phi(z,z')}{\partial y^+}\Big|_{z=z'},\quad d(z)=\frac{\partial^2\phi(z,z')}{\partial x^+\partial y^+}\Big|_{z=z'}.$$

Then, using property (1.6) we have

$$\frac{\partial\phi(z,z')}{\partial x}=-b(z)\phi(z,z'),\quad \frac{\partial\phi(z,z')}{\partial y}=-a(z)\phi(z,z'),$$

$$\frac{\partial^2\phi(z,z')}{\partial x\partial y}=d(z)\phi(z,z'),\quad d(z)=a(z)b(z)-\frac{\partial a(z)}{\partial x}\quad \text{and}$$

$$\frac{\partial a(z)}{\partial x}=\frac{\partial b(z)}{\partial y}\quad \text{for all } z'\leqslant z,\ z',z\in D.$$

In this situation, the function $\phi_2(z)$ can be calculated as follows:

$$\phi_2(z)=\left[\frac{\partial^2 E\,Y(z)}{\partial x\partial y}\right]^{\frac{1}{2}} = f(z)^{-1}.q(z),\ \text{where}$$

$$q(z)^2=\frac{\partial^2\Gamma(z)}{\partial x\partial y}+2a(z)\frac{\partial\Gamma(z)}{\partial x}+2b(z)\frac{\partial\Gamma(z)}{\partial y}+(4a(z)b(z)+2\frac{\partial b(z)}{\partial y})\Gamma(z).$$

These random fields can also be considered as aolutions of linear hyperbolic partial differential equations:

$$\frac{\partial^2 X}{\partial x \partial y} + a(z) \frac{\partial X}{\partial x} + b(z) \frac{\partial X}{\partial y} + c(z) X = q(z)\, \xi(z),$$

with the conditions $c = ab + \frac{\partial a}{\partial x}$ and $\frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$, where $\xi(z)$ is a white noise distributed on R^2_+.

We can attach a more precise meaning to these equations within the frame of the generalized stochastic processes theory.

As it is shown in [2] , the only stationary Gaussian Markov process $\{X_{st}\}_{st \in R^2}$ with non vanishing absolutely continuous variance function $\Gamma(z)$ is the two-parameter Ornstein-Uhlenbeck process, that is $X_{st} = \int_{R_{st}} \exp[-\alpha(s-x) - \beta(t-y)]\, dW_{xy}$, where $\alpha, \beta > o$.

Besides, using the rules of Itô's stochastic differential calculus in each coordinate, we obtain the following differential expressions for X:

$$(1.9)\quad \begin{aligned} d_1 X &= -b(z)\, X(z)\, dx + \phi_1(z)\, \alpha_1(z)\, W^{(1)}_y(dx), \\ d_2 X &= -a(z)\, X(z)\, dy + \phi_1(z)\, \alpha_2(z)\, W^{(2)}_x(dy). \end{aligned}$$

In these formulae $d_1 X = X(x+dx, y) - X(x,y)$ and $d_2 X = X(x, y+dy) - X(x,y)$ represent the differential increments in each coordinate, and $\alpha_1(z)$, $\alpha_2(z)$ are

$$\alpha_1(z) = \left[\int_{-\infty}^{y} \phi_2^2(x,\eta)\, d\eta\right]^{1/2}, \quad \alpha_2(z) = \left[\int_{-\infty}^{x} \phi_2^2(\xi, y)\, d\xi\right]^{1/2}.$$

The functions $W^{(1)}_y(A)$, $W^{(2)}_x(A)$ defined by

$$W^{(1)}_y(A) = \int_A \int_{-\infty}^{y} \frac{\phi_2(\xi,\eta)}{\alpha_1(\xi, y)}\, dW(\xi,\eta),$$

$$W^{(2)}_x(A) = \int_{-\infty}^{x} \int_A \frac{\phi_2(\xi,\eta)}{\alpha_2(x,\eta)}\, dW(\xi,\eta),$$

A being a Borel set in R, are ordinary Wiener measures when both variables y and x are fixed.

In particular, notice that $X(z)$ is a diffusion process in each coordinate.

In the same way, the rules of the two parameter differential stochastic calculus, developed in [1], and [5] give rise to the differential expression

$$\text{(1.10)}\quad \begin{aligned} dX = {} & q(z)\, dW(z) + d(z)\, X(z)\, dz - a(z)\, \phi_1(z)\, \alpha_1(z)\, dy\, W_y^{(1)}(dx) \\ & - b(z)\, \phi_1(z)\, \alpha_2(z)\, dx\, W_x^{(2)}(dy), \end{aligned}$$

where dX denotes the increment $X(x+dx,y+dy)-X(x,y+dy)-X(x+dx,y)+X(x,y)$.

This situation leads us to introduce some diffusion conditions for two parameter random Markov fields in order to consider the process given by (1.8) as a two parameter diffusion process and to provide some elements to study and characterize random Markov fields in the non Gaussian case.

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Let $X=\{X(z)\}_{z\in R_+^2}$ be a two parameter random Markov field, there exist a function

$$P(z_1,\ w_1,\ w,\ w_2;\ z_2,A),$$

defined for all $\bar{w} = (w_1,w,w_2)\in R^3$, z_1, $z_2\in R_+^2$, $z_1\leqslant z_2$, and Borel sets A, for which the following properties hold:

(1) $P(z_1,\bar{w};\ z_2,A)$ is a Borel function of $\bar{w}$,

(2) $P(z_1,\bar{w};\ z_2,.)$ is a probability for fixed $z_1,z_2,\bar{w}$,

(3) $P[X(z_2)\in A/X(x_1,y_2)=w_1, X(z_1)=w,\ X(x_2,y_1)=w_2] =$
$= P(z_1,\bar{w};\ z_2,A)$, w.p.1.

This function will be called the transition probability for X.

It also satisfies an equation of the Chapman-Kolmogorov type, that is, given a fixed point $\zeta=(\xi,\eta)$ such that $z_1\leqslant\zeta\leqslant z_1$, we have

$$\text{(2.1)}\quad P(z_1,\bar{w};\ z_2,A) = \iiint_{R^3} P(\zeta,\bar{v};z_2,A).P((x_1,\eta),(w_1,w_1',v);(\xi,\eta_2),dv_1).$$
$$.P((\xi,\eta_2),(v,w_2',w_2);(x_2,\eta,dv_2).P(z_1,(w_1',w,w_2');\zeta,dv).$$

This relation must be true for all w_1', $w_2' \in R$.

If we know the transition probabilities $P(z_1, \bar{w}; z_2, A)$ and the distributions of $\{X(o,y),\ o \leq y\}$ and $\{X_{(x,o)},\ o \leq x\}$, we can deduce all the finite dimensional distributions of X by using (2.1) as in the one parameter case.

If X vanishes on the axes, then X is a Markov process in each coordinate and its transition probabilities are:

(2.2) $P_y(x_1, w, x_2, A) = P[X(x_2 . y) \in A / X(x_1, y) = w] = P((x_1, o), (w, o, o); (x_2, y), A)$,

where y is fixed, and

(2.3) $P_x(y_1, w, y_2, A) = P[X(x, y_2) \in A / X_{(x, y_1)} = w] = P((o, y_1), (o, o, w); (x, y_2), A)$,

where x is fixed.

We assume in the following that X vanishes on the axes and has continuous sample paths. We also suppose the existence of a density $p(z_1, \bar{w}; z_2, u)$.

Let $z=(x,y)$ be a fixed point of R_+^2 and let us consider the increments

$$X(\Delta_1) = X(x+\Delta x, y) - X(z)$$
$$X(\Delta_2) = X(x, y+\Delta y) - X(z)$$
$$X(\Delta) = X(z') - X(x+\Delta x, y) - X(x, y+\Delta y) + X(z),$$

where $z' = (x+\Delta x, y+\Delta y)$ and $\Delta x > o$, $\Delta y > o$.

The stochastic differentials for the random Markov field X we have obtained in the Gaussian case have suggested us to introduce some conditions for X to be a two parameter diffussion process.

For the sake of simplicity we will omit truncations of the moments, stating only the usual conditions in strong sense.

<u>Definition 2.1.-</u> A random Markov field X is called a diffusion process in each coordinate iff:

(2.4) $$\lim_{\Delta x \to 0} \frac{E[X(\Delta_1)^k / X(z) = w]}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_R (w_2 - w)^k p_y(x, w, x+\Delta x, w_2)\, dw_2 = \begin{cases} a_1(z,w) & \text{if } k=1 \\ b_1(z,w) & \text{if } k=2 \\ o & \text{if } k>2 \end{cases}$$

(2.5) $$\lim_{\Delta y \to 0} \frac{E[X(\Delta_2)^k / X(z) = w]}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{\Delta y} \int_R (w_1 - w)^k p_x(y, w, y+\Delta y, w_1)\, dw_1 = \begin{cases} a_2(z,w) & \text{if } k=1 \\ b_2(z,w) & \text{if } k=2 \\ o & \text{if } k>2 \end{cases}$$

where $a_1, b_1, a_2 b_2$ are continuous functions, and represent the coefficients of diffusion and displacement in each coordinate.

Let us introduce the function

$$p(z,w;z',(w_1,u,w_2))=p(z,\bar{w};z',u).p_y(x,w,x+\Delta x,w_2).p_x(y,w,y+\Delta y,w_1)$$

for all $z \leq z'$, $\bar{w}=(w_1,w,w_2)\in R^3$, $u\in R$,

which verifies, using the Markov property, that

$$p(z,w;z',(w_1,u,w_2))=P\left[X(z')=u,\ X(x,y')=w_1,\ X(x',y)=w_2/X(z)=w\right],$$

w.p.1.

Definition 2.2.- A random Markov field X will be called a two parameter diffusion process iff it is a diffusion process in each coordinate and possesses the following properties:

$$\lim_{\Delta x,\ \Delta y\to o}\frac{E[X(\Delta)^k/X(z)=w]}{\Delta x\ .\ \Delta y}=\lim_{\Delta x,\ \Delta y\to o}\frac{1}{\Delta x.\Delta y}\iiint_{R^3}(u-w_1-w_2+w)^k$$

$$(2.6)\qquad p(z,w;z',(w_1,u,w_2))\ dw_1dudw_2=\begin{cases}a(z,w) & \text{if } k=1\\ b(z,w) & \text{if } k=2\\ o & \text{if } k>2\end{cases}.$$

$$\lim_{\Delta x,\Delta y\to o}\frac{E[X(\Delta)^kX(\Delta_1)^iX(\Delta_2)^j/X(z)=w]}{\Delta x\ .\ \Delta y}=\lim_{\Delta x,\Delta y\to o}\frac{1}{\Delta x.\Delta y}\iiint_{R^3}(u-w_1-w_2+w)^k.(w_1-w)^j.$$

$$(2.7)\qquad .(w_2-w)^i.p(z,w;z',(w_1,u,w_2))dw_1du\ dw_2=\begin{cases}c_1(z,w) & \text{if } k=i=1,\ j=o,\\ c_2(z,w) & \text{if } k=j=1,\ i=o,\\ d\ (z,w) & \text{if } k=i=j=1,\\ 0 & \text{if } k>1 \text{ and } i+j>1.\end{cases}$$

$a(z,w)$ and $b(z,w)$ will be called, respectively the two parameter coefficients of diffusion and displacement, and $c_1(z,w)$, $c_2(z,w)$, $d(z,w)$ will be called mixed diffusion coefficients. All of them are supposed continuous functions of (z,w).

Proposition 2.1.- If X is a two parameter diffusion process, we have

$$\lim_{\Delta x,\ \Delta y\to o}\frac{E[X(\Delta_1)^i\ X(\Delta_2)^j/X(z)=w]}{\Delta x\ .\ \Delta y}=\begin{cases}a_1(z,w).a_2(z,w) & \text{if } i=j=1,\\ a_1(z,w).b_2(z,w) & \text{if } i=1,\ j=2,\\ b_1(z,w).a_2(z,w) & \text{if } 1=2,\ j=1,\\ b_1(z,w).b_2(z,w) & \text{if } i=j=2,\\ 0 & \text{if } i>2 \text{ or } j>2.\end{cases}$$

In fact, the Markov property implies

$E[X(\Delta_1)^i \; X(\Delta_2)^j \,/\, X(z)=w] = E[X(\Delta_1)^i/X(z)=w] \,.\, E[X(\Delta_2)^j/X(z)=w]$.

Finally, the main result of this section is a "forward equation" for the evolution of the conditional probability density.

Theorem 2.1.- (Kolmogorov's Forward Equation). Let X be a two parameter diffusion process such that the transition probability density $p(z_1,\bar{w};z_2,u)$ has the partial derivatives

$\frac{\partial^2 p}{\partial x\,\partial y}$, $\frac{\partial p}{\partial u}$, $\frac{\partial^2 p}{\partial u^2}$, $\frac{\partial^3 p}{\partial u^3}$ and $\frac{\partial^4 p}{\partial u^4}$ continuous and also bounded in (z,u) for each fixed $z_1 \epsilon T$, $z_1 \leqslant z$, $\bar{w} \epsilon R^3$. We also assume that $\frac{\partial^2 p}{\partial x \partial y}$ is uniformly continuous in z with respect to (w_1,w_2) and u, that it is continuous in (w_1,w_2), and bounded

$$\left|\frac{\partial^2 p}{\partial x\,\partial y}\right| \leqslant g(u) \text{ for fixed } z_1 \text{ and } w,$$

g(u) being a Lebesgue integrable function.

Then we have,

$$\frac{\partial^2 p}{\partial x\,\partial y} = -\frac{\partial (p\,M_1)}{\partial u} + \frac{1}{2}\frac{\partial^2 (p\,M_2)}{\partial u^2} - \frac{1}{2}\frac{\partial^3 (p\,M_3)}{\partial u^3} + \frac{1}{4}\frac{\partial^4 (p\,M_4)}{\partial u^4} \tag{2.8}$$

where the infenitessimal moments of order i, $M_i(z,u)$, i=1,2,3,4, are given by

$$M_i(z,u) = \sum_{\substack{\alpha+\beta+\gamma=i \\ \alpha,\beta,\gamma\,\epsilon\,N}} \lim_{\Delta x,\Delta y \to o} \frac{E[X(\Delta)^\gamma \; X(\Delta_1)^\alpha \; X(\Delta_2)^\beta / X(z)=u]}{\Delta x \,.\, \Delta y}, \tag{2.9}$$

that is,

$M_1 = a$

$M_2 = b + 2c_1 + 2c_2 + 2a_1 a_2$

$M_3 = a_1 b_2 + a_2 b_1 + 2d$

$M_4 = b_1 b_2$.

We also suppose the terms in the limit (2.9) to be bounded by a finite constant k_i independent of (z,u).

Proof: Let $f:R\to R$ be infinitely differentiable with compact support, and, for fixed $(z,\bar{w})$, consider the function defined by
$\varphi(z)=E[f(X(z)) \;/\; X(z_1)=w,\ X(x_1,y)=w_1,\ X(x,y_1)=w_2]=\int_R f(u)\,p(z_1,\bar{w};z,u)\,du.$

We know there exist

(2.10) $\dfrac{\partial^2\varphi}{\partial x\,\partial y} = \displaystyle\int_R f(u)\,\frac{\partial^2 p(z_1,\bar{w};z,u)}{\partial x\,\partial y}\,du,$

and $\dfrac{\partial^2\varphi}{\partial x\,\partial y}$ is continuous and bounded in (w_1,w_2,z) for fixed (z_1,w).

Therefore, diffusion conditions in each coordinate imply

(2.11) $$\frac{\partial^2\varphi}{\partial x\,\partial y}(w_1,w_2,z)=\lim_{\Delta x,\,\Delta y\to o}\iint_{R^2}\frac{\partial^2\varphi}{\partial x\,\partial y}(w_1',w_2',z)\,p_{y_1}(x,w_2;x',w_2')\,p_{x_1}(y,w_1;y',w_1')\,dw_1'\,dw_2'\ ,$$
where $x'=x+\Delta x$ and $y'=y+\Delta y$.

Using the mean value theorem,

$$\frac{\varphi(z')-\varphi(x',y)-\varphi(x,y')+\varphi(z)}{\Delta x\cdot\Delta y}(w_1',w_2') = \frac{\partial^2\varphi}{\partial x\,\partial y}(w_1',w_2',z''),$$
where $z\leqslant z''\leqslant z'$.

Uniform continuity of $\dfrac{\partial^2\varphi}{\partial x\,\partial y}$ in z gives

$$\lim_{\Delta x,\Delta y\to o}\iint_{R^2}\Big[\frac{\partial^2\varphi}{\partial x\,\partial y}(w_1',w_2',z)-\frac{\partial^2\varphi}{\partial x\,\partial y}(w_1',w_2',z'')\Big]\,p_{y_1}(x,w_2;x',w_2')\,p_{x_1}(y,w_1;y',w_1')\,dw_1'\,dw_2' = 0,$$
then from (2.11) we obtain

(2.12) $$\lim_{\Delta x,\Delta y\to o}\frac{1}{\Delta x\Delta y}\iint_{R^2}\big[\varphi(z')-\varphi(x'y)-\varphi(x,y')+\varphi(z)\big]\,(w_1',w_2').\,p_{y_1}(x,w_2;x',w_2').\,p_{x_1}(y,w_1;y',w_1')\,dw_1'\,dw_2' = \frac{\partial^2\varphi}{\partial x\,\partial y}(w_1,w_2,z).$$

Using Chapman-Kolmogorov's equation, we write

$$[\varphi(z')-\varphi(x',y)-\varphi(x,y')+\varphi(z)] \quad (w'_1,w'_2) =$$

$$=\int_R f(u')p(z_1,(w'_1,w,w'_2);z',u')du'-\int_R f(u_2)p(z_1,(w_1,w,w'_2);(x',y),u_2)du_2-$$

$$-\int_R f(u_1)p(z_1,(w'_1,w,w_2);(x,y'),u_1)du_1+\int_R f(u)p(z_1,\bar{w};z,u)du =$$

$$=\int_R f(u')[\iiint_{R^3} p(z,(u_1,u,u_2);z',u')p((x_1,y),(w'_1,w_1,u);(x,y'),u_1)$$

$$p((x,y_1),(u,w_2,w'_2);(x',y),u_2).p(z_1,\bar{w};z,u)du_1du\ du_2]du' -$$

$$-\int_R f(u_2)[\int_R p((x,y_1),(u,w_2,w'_2);(x',y),u_2)p(z_1,\bar{w};z,u)du]\ du_2-$$

$$-\int_R f(u_1)[\int_R p((x_1,y),w'_1,w_1,u);(x,y'),u_1)p(z_1,\bar{w};z,u)du]\ du_1+$$

$$+\int_R f(u)p(z_1,\bar{w};z,u)\ du.$$

If we substitute in (2.12) taking into account that

$p_y(x,u;x',u_2) =\int_R p((x,y_1),(u,w_2,w'_2);(x',y),u_2)\ p_{y_1}(x,w_2;x',w'_2)dw'_2$, and

$p_x(y,u;y',u_1) =\int_R p((x_1,y),(w'_1,w_1,u);(x,y')u_1)p_{x_1}(y,w_1;y',w'_1)dw'_1$,

we have

$$\frac{\partial^2\varphi}{\partial x\partial y}(w_1,w_2,z)=\lim_{\Delta x,\Delta y\to o}\frac{1}{\Delta x\ \Delta y}\int_R E[f(X(z')-f(X(x',y))-f(X(x,y'))+$$

$$+f(X(z))\ /\ X(z)=u]\ .\ p\ (z_1,\bar{w};z,u)du.$$

Using a Taylor expansion of f and the hypotheses for the moments $M_i(z,u)$, we obtain

$$\frac{\partial^2\varphi}{\partial x\partial y} = \int_R [M_1(z,u)f'(u)+\frac{1}{2}M_2(z,u)f''(u)+\frac{1}{2}M_3(z,u)f'''(u)+\frac{1}{4}M_4(z,u)f^{IV}(u)]$$

$$p(z_1,\bar{w};\ z,\ u)du.$$

Finally, integrating by parts and using $f^{(k)}(\pm\infty)=0$, we conclude the theorem proof by means of the identification with formula (2.10).□

- 3 -

Let $X=\{X(z)\}_{z\in R^2_+}$ be a two-parameter diffusion process and $f(z,u)$ a monotone function in $u\in R$ for all $z\in R^2_+$. Let $g(z,w)$ denote the inverse function of $f(z,u)$.

Then, the random field $Y(z)=f(z,X(z))$ is also a Markov random process with transition probability $\bar{P}(z_1,\bar{w},z_2,A)$ related to the one of X by

$$\bar{P}(z_1,\bar{w};z_2,A)=P(z_1,(g((x_1,y_2),w_1),g(z,w),g((x_2,y_1),w_2));z_2,g(z_2,A)).$$

If f is differentiable and bounded enough (i. e.,

$\frac{\partial^4 f}{\partial u^4}$, $\frac{\partial^3 f}{\partial x\,\partial u^2}$, $\frac{\partial^3 f}{\partial y\,\partial u^2}$, $\frac{\partial^2 f}{\partial x\,\partial y}$ and f is continuous and bounded), then Y is a two parameter diffusion process. Boundness hypotheses can be eliminated weaking diffusion conditions by means of truncations.

The coefficients for the diffusion process Y in each coordinate are given by

$$\bar{a}_1(z,w)=\frac{\partial f}{\partial x}(z,u)+a_1(z,u)\frac{\partial f}{\partial u}(z,u)+\frac{1}{2}b_1(z,u)\frac{\partial^2 f}{\partial u^2}(z,u)=D_1(f)(z,u),$$

$$\bar{b}_1(z,w)=b_1(z,u)\left[\frac{\partial f}{\partial u}(z,u)\right]^2,$$

$$\bar{a}_2(z,w)=\frac{\partial f}{\partial y}(z,u)+a_2(z,u)\frac{\partial f}{\partial u}(z,u)+\frac{1}{2}b_2(z,u)\frac{\partial^2 f}{\partial u^2}(z,u)=D_2(f)(z,u),$$

$$\bar{b}_2(z,w)=b_2(z,u)\left[\frac{\partial f}{\partial u}(z,u)\right]^2,$$

where $u=g(z,w)$ and D_1,D_2 denote the diffusion operators in each coordinate. Using a Taylor expansion of f we can also calculate two parameter coefficients:

$$\bar{a}(z,w)=D_1(\frac{\partial f}{\partial y}) +D_2(\frac{\partial f}{\partial x}) - \frac{\partial^2 f}{\partial x\partial y} +M_1\frac{\partial f}{\partial u} + \frac{1}{2} M_2\frac{\partial^2 f}{\partial u^2} + \frac{1}{2} M_3\frac{\partial^3 f}{\partial u^3} + \frac{1}{4} M_4\frac{\partial^4 f}{\partial u^4},$$

$$\bar{b}(z,w)=b.[\frac{\partial f}{\partial u}]^2 +b_1b_2[\frac{\partial^2 f}{\partial u^2}]^2+ 2d.\frac{\partial f}{\partial u}.\frac{\partial^2 f}{\partial u^2},$$

$$\bar{c}_1(z,w)= \frac{\partial f}{\partial u}.[b_1.D_2(\frac{\partial f}{\partial u}) + c_1.\frac{\partial f}{\partial u} +d.\frac{\partial^2 f}{\partial u^2}],$$

$$\bar{c}_2(z,w)= \frac{\partial f}{\partial u}.[b_2.D_1(\frac{\partial f}{\partial u}) + c_2.\frac{\partial f}{\partial u} + d.\frac{\partial^2 f}{\partial u^2}],$$

$$\bar{d}(z,w)=d.[\frac{\partial f}{\partial u}]^3 +b_1b_2[\frac{\partial f}{\partial u}]^2 . \frac{\partial^2 f}{\partial u^2},$$

where we have omitted for simplicity the dependence with respect to (z,u) in the second term.

For example, given the coefficients of the two parameter Wiener process $W=\{W(z)\}_{z\in R_+^2}$ as a diffusion process:

$$a_1= a_2 = 0,\ b_1= y,\ b_2= x,$$

$$b = 1,\ a = c_1 = c_2 = d = 0,$$

we can calculate by means of the preceding formulas the coefficients of $X(z)=f(z, W(z))$:

$$\bar{a}_1= \frac{\partial f}{\partial x} + \frac{1}{2} y \frac{\partial^2 f}{\partial u^2} = D_1(f),\ \bar{b}_1= y.[\frac{\partial f}{\partial u}]^2,$$

$$\bar{a}_2= \frac{\partial f}{y} + \frac{1}{2} x \frac{\partial^2 f}{\partial u^2} = D_2(f),\ \bar{b}_2= x.[\frac{\partial f}{\partial u}]^2,$$

$D_2)(f)$,

$, [\frac{\partial^2 f}{\partial u^2}]^2,$

These results can be compared with the differential representation of the random field X(z) obtained by using the rules of the two parameter stochastic differential calculus [4]:

$$d_1(X(z) = D_1(f)\ dx + \sqrt{y}\ .\ \frac{\partial f}{\partial u}\ W_y^{(1)}(dx),$$

$$d_2(X(z) = D_2(f)\ dy + \sqrt{x}\ .\ \frac{\partial f}{\partial u}\ W_x^{(2)}(dy),$$

$$d\ X(z) = \frac{\partial f}{\partial u}\ .\ dW + \sqrt{xy}\ \frac{\partial^2 f}{\partial u^2}\ W_y^{(1)}(dx)\ W_x^{(2)}(dy) + \ y.D_2(\frac{\partial f}{\partial u})\ dy\ W_y^{(1)}(dx) +$$

$$+\sqrt{x}.D_1(\frac{\partial f}{\partial u})dx\ W_x^{(2)}(dy) + (D_1 o\ D_2)(f)\ d\ x\ dy,$$

where $W_y^{(1)}(x) = \frac{W(x,y)}{y}$, $W_x^{(2)}(y) = \frac{W(x,y)}{x}$ are Brownian motions in x and y respectively.

Finally, the Gaussian random function $X(z)=\emptyset_1(z)\int_o^x\int_o^y \emptyset_2(\alpha)\ d\ W(\alpha)$, z = (x,y), introduced in Section 1, is also a two parameter diffusion process with coefficients

$$a_1= -b\ u\quad,\quad b_1= \emptyset_1^2\alpha_1^2\quad,$$

$$a_2= -a\ u\quad,\quad b_2= \emptyset_1^2\alpha_2^2\quad,$$

$$a = d\ u,\ b=q^2,\ c_1= -a\ \emptyset_1^2\alpha_1^2\quad,\ c_2= -b\ \emptyset_1^2\alpha_2^2\quad,\ d = 0.$$

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