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AN EXACT PROCEDURE FOR 2x2x2 CONTINGENCY TABLES

by

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If n independent observations, categorized according to three schemes with two categories in each scheme, have been taken, it is customary to summarize the data in a 2x2x2 contingency table.

Let n_{ijk} be the entries in the table, and put $n_{i..} = \sum_{jk}\sum n_{ijk}$, $n_{.j.}$ and $n_{..k}$ analogously. Clearly, $n = n_{1..} + n_{2..} = n_{.1.} + n_{.2.} = n_{..1} + n_{..2}$ holds. Let p_{ijk}, $p_{i..}$, $p_{.j.}$, and $p_{..k}$ be the corresponding probabilities. Under the null hypothesis that the three schemes are independent, the probabilities in the body of the table are products of the marginal totals:

$$H_o: p_{ijk} = p_{i..} \cdot p_{.j.} \cdot p_{..k} .$$

If the sample size n is small, a conditional test based on exact distributions can be constructed as follows (this approach is well known in the 2x2 case. cf. Lindgren (1968), p. 425, e.g.).

First notice that for given marginals $n_{1..}$, $n_{.1.}$, $n_{..1}$, a 2x2x2 table is completely specified if the four entries n_{111}, n_{121}, n_{122}, n_{211} are known. Under the null hypothesis of independence, we have

$$P(n_{111}=h_{111}, n_{121}=h_{121}, n_{122}=h_{122}, n_{211}=h_{211} \mid n_{..1}=r, n_{.1.}=s, n_{1..}=t)$$

$$= \frac{P(n_{ijk}=h_{ijk},\ i,j,k=1,2)}{P(n_{..1}=r, n_{.1.}=s, n_{1..}=t)}$$

$$= n! \prod_{i,j,k} \frac{p_{ijk}^{h_{ijk}}}{h_{ijk}!} \quad \frac{1}{\binom{n}{r}\binom{n}{s}\binom{n}{t} p_{..1}^{r}\, p_{..2}^{(n-r)}\, p_{.1.}^{s}\, p_{.2.}^{(n-s)}\, p_{1..}^{t}\, p_{2..}^{(n-t)}}$$

$$= \frac{n!}{\binom{n}{r}\binom{n}{s}\binom{n}{t} \prod_{i,j,k} h_{ijk}!}$$

$$= \binom{n}{h_{111},\ldots,h_{222}} \Big/ \binom{n}{r}\binom{n}{s}\binom{n}{t}.$$

So, for any given set of marginals $n_{1..}, n_{.1.}, n_{..1}$, we just have to fix a region of tables having these marginals in such a way that the conditional probability of this region does not exceed some specified α. Because of

$$\begin{aligned} & P_{H_o}(H_o \text{ is rejected}) \\ &= E[P_{H_o}(H_o \text{ is rejected} \mid n_{1..}, n_{.1.}, n_{..1})] \\ &\leq E[\alpha] = \alpha , \end{aligned}$$

the size of the type I error for this test does not exceed α.

Of course, the shape of such a rejection region depends heavily on the specification of the alternative hypothesis H_1. If, for instance, second order interactions are conjectured, i.e.

$$H_1: \frac{p_{111}p_{221}}{p_{211}p_{121}} \neq \frac{p_{112}p_{222}}{p_{212}p_{122}} ,$$

it is near at hand to reject the null hypothesis if

$$\frac{n_{111}n_{221}n_{212}n_{122}}{n_{112}n_{222}n_{211}n_{121}}$$

differs from 1 considerably.

Obviously, the above procedure can be performed easily by using a simple computer program.

BIBLIOGRAPHY

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