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BRIEF SURVEY OF SEMIGROUP THEORY AND ITS APPLICATIONS TO EVOLUTION PROBLEMS

by

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0. Introducción

Let Σ be a "physical" system, such as a deformable continuum, a quantum-mechanical system, a rigid conductor of heat, etc. The system will be characterized by a suitable state vector $u(\dot{x},t)$, with x a set of space-like variables, and t a time-like parameter. Usually, the state u belongs to a space of functions X -in general, a Banach space- and the evolutionary properties of Σ are summarized by a set of operations A which transform u into another element of the state space X. In other words, the operator $A:D\times R \to X$, where D is a subset of the functional space X, will be the mathematical counterpart of those physical phenomena which make the state u undergo a change as time goes on. In general, we shall have

$$\frac{du}{dt} = \dot{u}(t) = A(u(t),\ t) \qquad (1)$$

a non-linear, non-autonomons, abstract evolution equation. For each $(v,s) \in D \times R$ the <u>Cauchy Problem</u> is to find a map $u(\cdot) \in C([s,\infty);\ D) \cap C^1([s,\infty);\ X)$ such that (1) holds on $[s,\infty)$ and $u(s) = v$, where v

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is the prescribed initial value (1). If these solution maps exist and are unique, we can define the family of operators

$$U(t,s) : D \to D, \quad t \in R^+, \ s \in R,$$

that map the state which at time s was at v to the state at time t+s. Such a family is called a semi-evolution system with generator A. It is obvious that we have the properties

$$U(0,s) = I \text{ (identity)}$$
$$U(t+\tau,s) = U(t,s+\tau)\ U(\tau,s)$$

for $t,\tau \in R^+$, $s \in R$. In the special case when the operators $U(t,s)$ are independent of s, the semi-evolution system defines a semi-group $\{U(t)\}$, $t \in R^+$. In the above R^+ denotes the nonnegative reals.

From the standpoint of applications, a fundamental problem in differential equations is to find whether a given mapping A generates a semigroup of operators. Also, we can ask if a given semigroup of operators has a generator.

In the first part of this Lecture we shall see that for a general Banach space with $\{U(t)\}$ linear, for all $t \in R^+$ -and, therefore, A linear- the Hille-Yosida theory gives a complete answer. In part 2, we shall see that for non-linear semigroups in a Hilbert space we can still have a complete characterization. For non-linear semigroups in Banach spaces not known necessary and sufficient conditions are available.

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(1) The vector state u(x,t) must usually satisfy some requirements when x belongs to the boundary of the physical system Σ, and these boundary conditions are generally taken into account via the subset D.

1. Linear semigroups.

We assume from now on that X is a Banach space over the field K of real or complex numbers, and that $|\cdot|$ is the norm on X.

The basic definition of a semigroup embodies the idea that a linear evolution equation

$$\dot{u}(t) = \frac{du}{dt} = A\,u(t),$$

where A is an operator in the Banach space X, posesses, for initial data in the domain of A, D(A), unique solutions which vary continuously in X as the initial data vary in the X topology.

Definitions. A family $\{U(t)\}$, $t \geq 0$, of bounded linear mappings from X into X is said to be a strongly continuous semigroup, or semigroup of class (C_o), if

(a) $U(0) = I$, $U(t+s) = U(t)\,U(s)$ for all $t,s \geq 0$;

(b) for each $v \in X$ the map $t \to U(t)v$ is continuous from $[0,\infty)$ into X (2).

The generator of the semigroup $\{U(t)\}$, $t \geq 0$, is the operator A with domain

$$D(A) = \{ v \in X \mid \lim_{h \downarrow o} h^{-1}(U(h)-I)v \ \text{ exists in } X\}$$

and value

$$Av = \lim_{h \downarrow o} h^{-1}(U(h)-I)v = \lim_{h \downarrow o} A_h v.$$

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(2) For the definition of strong and weak continuity and differentiability see, for instance, [1, 24, 17]. Continuity and differentiability shall be understood in the strong sense unless otherwise stated.

By using the uniform boundedness theorem [17] and properties (a) and (b), it is easy to show [2, p. 277] that a semigroup of class (C_o) satisfies the condition

$$\| U(t)\| \leqslant Me^{\beta t}, \qquad t \in [0,\infty)$$

with constants $M > 0$ and $\beta < \infty$. $\|\cdot\|$ denotes the norm on $B(X)$, which is the family of all bounded linear operators with domain X and range in X. If $\| U(t)\| \leqslant e^{\beta t}$ the semigroup is called <u>quasi-contractive</u> and <u>contractive</u> if, furthermore, $\beta = 0$. We shall denote by $G(X,M,\beta)$ the set of generators of type M,β. If $v \in X$, it is easy to check that $\sup|e^{-\beta t} U(t)v|$ is a norm on X equivalent to the given norm $|\cdot|$, so that, in this sense, quasi-contractive semigroups become fundamental.

The next result establishes several important properties of linear semigroups:

<u>Proposition 1.</u> Suppose that A is the generator of the strongly continuous semigroup $\{U(t)\}$, $t \geqslant 0$. Then, we have the following properties:

(i) $U(t)\, D(A) = \{U(t)v \mid v \in D(A)\} \subset D(A) \qquad \forall\ t \geqslant 0;$

(ii) $U(t)\, Av = AU(t)v \qquad \forall\ v \in D(A), \quad \forall\ t \geqslant 0;$

(iii) if $v \in D(A)$, the map $t \to U(t)v$ is continuously differentiable on $[0,\infty)$ and it satisfies

$$\frac{d}{dt} U(t)v = AU(t)v,$$

namely, $U(t)v$ is a solution of the initial value problem $\dot{u}(t) = Au(t)$, $u(0) = v$;

(iv) A is a closed densely defined linear operator on X;

(v) if $u:[0,\infty) \to D(A)$ is a differentiable map, such that $\dot{u}(t) = Au(t)$ for all $t \geqslant 0$, then $u(t) = U(t)u(0)$ for all $t \geqslant 0$, i.e., the existence of a solution to the Cauchy problem implies its uniqueness.

The proof of (ii) is a straightforward consequence of the definition of the generator and the boundedness and semigroup properties of $U(t)$; then, (i) follows immediately. Obviously, the facts

$$\lim_{h\downarrow o}\{h^{-1}(U(t)v-U(t-h)v)-U(t)Av\}=\lim_{h\downarrow o} U(t-h)(A_h v-Av)+\lim_{h\downarrow o}[U(t-h)-U(t)]Av=0$$

$$\lim_{h\downarrow o}\{h^{-1}(U(t+h)v-U(t)v)-U(t)Av\}=\lim_{h\downarrow o} U(t)[A_h v-Av]=0,$$

together with the continuity of $t \to U(t)Av$, imply (iii). Result (v) is a consequence of

$$|(f(t+h)-f(t))h^{-1}| \leqslant M\exp[\beta(t_o-t-h)]\,|h^{-1}[u(t+h)-U(h)u(t)| \to 0$$

as $h \to 0$, where $f(t) \equiv U(t_o-t)u(t)$, $t_o > 0$ fixed, $t\in[0,t_o)$. Finally, if we define $v_t=t^{-1}\int_o^t U(s)v\,ds$ (3), for $v \in X$ and $t>0$, it is easy to see that we have, for $h>0$,

$$A_h v_t=t^{-1}\{h^{-1}\int_t^{t+h} U(s)v\,ds-h^{-1}\int_o^h U(s)v\,ds\} \to t^{-1}[U(t)v-v]= Av_t$$

as $h\downarrow 0$, whence $v_t \in D(A)$ and, in particular,

$$U(t)v-v = A\int_o^t U(s)v\,ds \quad \text{for all} \quad v\in X,\ t\geqslant 0; \qquad (2)$$

now, since $v_t \to v$ as $t\downarrow 0$, $D(A)$ will be dense in X, and, moreover, if $\{v_n\} \subset D(A)$, $n=1,2,\ldots,v_n \to v$, and $Av_n \to u$ as $n\to\infty$, we have by (iii)

$$U(t)v_n-v_n= \int_o^t U(s)Av_n\,ds,\ n\geqslant 1,\ t>0,$$

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(3) A brief but sufficient outline of integral calculus for abstract functions can be found in [1] .

which proves that $Av = u$ by passing to the limit as $n\to\infty$, dividing the result by t^{-1} and letting $t\downarrow 0$; thus, result (iv) follows.

We recall that if A is a linear operator on X then the <u>resolvent set</u> of A is the set

$$\varphi(A) = \{\lambda \in K \mid (\lambda I-A)^{-1} \text{ exists and is in } B(X)\}.$$

The <u>resolvent</u> of A is the map $R(A): \varphi(A) \to B(X)$ defined by $R(A;\lambda) = (\lambda I-A)^{-1}$ for all $\lambda \in \varphi(A)$.

We now state the fundamental theorem of Hille-Yosida.

<u>Theorem 1.</u> A necessary and sufficient condition for a closed, densely defined linear operator A to be an element of $G(X,M,\beta)$ is that, for every real $\lambda>\beta$, [4]

(a) $\lambda \in \varphi(A)$

(b) $\| (\lambda I-A)^{-n} \| \leqslant \dfrac{M}{(\lambda-\beta)^n}$, $n = 1,2,\ldots$.

<u>Outline of Proof.</u>

<u>Necessity.</u> It is simple to verify that $e^{-\lambda t}U(t)$ is a semigroup with generator $A-\lambda I$, so by (2)

$$e^{-\lambda t}\,U(t)v-v = (A-\lambda I)\int_0^t e^{-\lambda s}\,U(s)v\,ds, \text{ for } t\geqslant 0,\ v\in X,$$

and by (iii) of Proposition 1

$$e^{-\lambda t}\,U(t)v-v = \int_0^t e^{-\lambda s}\,U(s)(A-\lambda I)v\,ds, \text{ for } t\geqslant 0,\ v\in D(A).$$

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(4) If condition (b) holds true for $|\lambda|>\beta$, then we get a group, namely, $U(t)$ is defined forward and backward in time.

If $\lambda>\beta$, we let $t \to \infty$ in the above equations, and noting that $A-\lambda I$ is closed, we obtain the Laplace transform relation

$$v = (\lambda I-A) \int_0^\infty e^{-\lambda t} U(t)vdt, \qquad \forall v \in X \tag{3}$$

and

$$v = \int_0^\infty e^{-\lambda t} U(t)(\lambda I-A)vdt, \qquad \forall v \in D(A).$$

Hence, $(\lambda I-A): D(A) \to X$ is one-to-one and onto, and, furthermore, $|(\lambda I-A)^{-1}v| \leqslant M(\lambda-\beta)^{-1}|v|$. Using now the resolvent identity $R(A;\lambda) - R(A;\mu) = (\mu-\lambda)\ R(A;\lambda)\ R(A;\mu)$, dividing both sides of this identity by $\lambda-\mu$, taking $\mu \to \lambda$ and applying induction, we get $(d/_{d\lambda})^{n-1}(\lambda I-A)^{-1}=(-1)^{n-1}(n-1)!\,(\lambda I-A)^{-n}$, which can be used to differentiate (3), and together with

$$\int_0^\infty e^{-\lambda t}\, t^{n-1}\, dt = (n-1)!\,/\,\lambda^n,$$

gives the estimate (b).

Sufficiency (Existence and, by Proposition 1 (v), Uniqueness Theorem).

Since $\bar{U}(t) \equiv e^{-\beta t}\, U(t)$ is a semigroup with generator $A - I\beta$, $\|\bar{U}(t)\| \leqslant M$, and, if we can prove the existence of $\bar{U}(t)$, then $U(t)$ will satisfy the conditions of the theorem. Hence, we can assume $\beta=0$ without loss of generality, and estimate (b) becomes $\|(I-\mu A)^{-n}\| \leqslant M$ with $\mu = \frac{1}{\lambda} > 0$.

If we set $U_n(t) \equiv (I- \frac{t}{n} A)^{-n}$, we have $\|U_n(t)\| \leqslant M$ for all n,

$$\frac{d}{dt}[\,U_n(t)v\,] = A(I- \frac{t}{n} A)^{-1}\, U_n(t)v, \tag{4}$$

and $U_n(t)v \to U_n(0)v=v$ as $t\downarrow 0$ -note that $(I-\mu A)^{-1}v-v=\mu(I-\mu A)^{-1}Av$ for $v\in D(A)$, hence $(I-\mu A)^{-1}v\to v$ as $\mu\downarrow 0$ on X-. Now,

$$U_n(t)v-U_m(t)v = \lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{t-\varepsilon}\frac{d}{ds}\,[U_m(t-s)U_n(s)v]\,ds$$

$$= \int_0^t [(\frac{s}{n})-\frac{(t-s)}{m}]\,(I-\frac{t-s}{m}A)^{-m-1}(I-\frac{s}{n}A)^{-n-1}A^2v\,ds,\ \forall\ v\in D(A^2)$$

and, hence, we can easily obtain that $U_n(t)v$ form a Cauchy sequence[5] and $\lim_{n\to\infty} U_n(t)v = U(t)v$, for all $v\in X$, where the denseness of $D(A^2) = (I-A)^{-1}D(A)$ in X and the uniform boundedness of $U_n(t)$, $n\geqslant 1$, are used. Now, it is easy to check that $U(t)$ has the desired properties for a strongly continuous semigroup[6]. Finally, integrating (4) and taking limits, we obtain

$$U(t)v = v+t\int_0^t U(s)Av\,ds, \qquad v\in D(A), \tag{5}$$

which implies $v\in D(A')$, $A'v = Av$ and A' is an extension of A,A' being the generator of $U(t)$. But, by the first part of the theorem, $(I-A')^{-1}$ exists and $I-A'$ is one-to-one and onto, thus $D(A) = D(A')$.

<u>Remark 1.</u> If $A\in B(X)$, the operator $U(t)$ can be simple defined as

$$U(t)= e^{tA} = \sum_{n=o}^{\infty}\frac{(tA)^n}{n!} \qquad \forall\ t\geqslant 0$$

which converges in the uniform topology -that is, in the sense of the

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(5) $\lim U_n(t)v$ exists uniformly in t, in any finite interval.

(6) In proving $U(t+\tau)=U(t)U(\tau)$ we can use (5), which shows that $U(t)v$ is differentiable, together with the uniqueness result of Proposition 1. For details of this sufficiency proof see, for instance, [3, Chap. IX].

norm of B(X)-. For partial differential equations, A will be unbounded in general, and in the sufficiency part of the proof of the above theorem we have used an operator analogue to $e^x = \lim_{n\to\infty} (1-\frac{x}{n})^n$.

<u>Corollary 1.</u> A linear operator A on the Banach space X has a closure $\bar{A}$ which is a member of $G(X,1,\beta)$ iff (i) D(A) is dense, (ii) $\lambda I-A$ has dense range for λ sufficiently large, and (iii) $|(\lambda I-A)v| \geqslant (\lambda-\beta)\ |v|$.

<u>Corollary 2. (Lumer-Phillips Theorem).</u> A linear operator on a Hilbert space H has a closure $\bar{A}\in G(H,1,\beta)$ iff (i) D(A) is dense in H, (ii) for λ sufficiently large $\lambda I-A$ has dense range, and (iii) $\mathrm{Re}(Av,v) \leqslant \beta(v,v)$ for all $v\in D(A)$, where $(\cdot,\cdot)$ is the inner product on H.

Corollary 2 is a consequence of Corollary 1 and $|\mathrm{Re}\ ((\lambda-A)v,v)| \geqslant (\lambda-\beta)|v|^2$ together with Cauchy-Schwarz inequality, and for the converse $\mathrm{Re}(v,U(t)v)\leqslant |v|^2$, which implies $\mathrm{Re} \lim_{t\downarrow o} t^{-1}(v,U(t)v-v) = \mathrm{Re}(v,\bar{A}v) \leqslant 0$ (recall that we can take $\beta=0$ without loss of generality). Necessity in Corollary 1 follows easily from Theorem 1; for the proof of sufficiency see [4].

In order to illustrate the previous abstract results, we give a few examples.

<u>Example 1.</u> As a simple example we shall consider the initial boundary value problem for the propagation of heat in a rigid slab:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0< x< 1, \qquad t> 0$$

$$u(0,t) = u(1,t) = 0 \qquad \forall\ t >0 \qquad (6)$$

$$u(x,0) = u_o(x) \qquad 0 <x< 1,$$

where u(x,t) is the temperature of the slab. In order to write (6) as an abstract initial-value problem, we introduce the Banach space $(X, |\cdot|)$, where $X=\{u \mid u \in C[0,1],\ u(0)=u(1)=0\}$, and $|u| = \sup|u(x)|$ when $x \in [0,1]$ (7). We shall see, by direct calculation, that the operator $A = d^2/dx^2$ with domain $D(A) = \{u \in X \mid Au \in X\}$ is a member of $G(X,1,0)$.

First of all, we observe that $C_o^\infty[0,1] \subset D(A) \subset X$, and, since $C_o^\infty[0,1]$ is dense in X, we have $\overline{D(A)} = X$. Secondly, we find the solution $u \in D(A)$ of $(\lambda I - A)u = v$, for $\lambda > 0$ and $v \in X$ given; it is easy, though rather tedious, to see that this solution, with $u(0) = u(1) = 0$, has the form

$$u(x) = (\lambda I - A)^{-1} v(x) = \frac{1}{2\mu \, sh_\mu} [\, f(\mu) v(x) + g(\mu) v(x) \,], \quad \mu = \sqrt{\lambda} \ ,$$

where

$$f(\mu) v(x) \ = \ - \int_o^1 ch[\, \mu(x+y-1) \,] v(y) \ dy,$$

and

$$g(\mu) v(x) = \int_x^1 ch[\, \mu(1+x-y) \,] v(y) dy + \int_o^x ch[\, \mu(1-x+y) \,] v(y) dy.$$

It is obvious that $u \in D(A)$. Furthermore, we get, via an easy estimate,

$$|u| = | \ (\lambda I - A)^{-1} v| \leqslant \frac{1}{\lambda} \ |v| \ , \qquad \lambda > 0, \quad v \in X,$$

and, hence, $(\lambda I - A)^{-1} \in B(X)$ with $\| (\lambda I - A)^{-1} \| \leqslant \frac{1}{\lambda}$, $\lambda > 0$. Finally, $A = -(\lambda I - A) + \lambda I$ is closed, since $(\lambda I - A)^{-1}$ is bounded, and, therefore, $(\lambda I - A)^{-1}$, $(\lambda I - A)$ are closed. Then, by Theorem 1, A is the generator of a contraction semigroup which solves strongly the Cauchy problem.

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(7) The difference between $|u|$ and $|u(x)|$ is self-explanatory.

Example 2. Assume now that we have a "nice" region $\Omega \subset R^n$ with boundary $\partial\Omega$, and we want to solve the parabolic heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta u(x,t), \quad x \in \Omega, \quad t > 0, \quad \Delta \text{ laplacian operator,}$$

with initial condition $u(x,0) = u_0(x)$, $x \in \Omega$, and boundary conditions, either

$$u(x,t)=0 \text{ for } x \in \partial\Omega, \quad t > 0 \quad \text{(Dirichlet)}, \qquad \text{(D)}$$

or

$$\frac{\partial u}{\partial n}(x,t)=0 \text{ for } x \in \partial\Omega, \quad t > 0 \quad \text{(Neumann)}, \qquad \text{(N)}$$

where n is the unit interior normal. Let $X = L_2(\Omega)$ be the state space and

$$A_D = \Delta, \quad D(A_D) = \{u \in L_2(\Omega) \mid u \in H^1_0(\Omega), \ \Delta u \in L_2(\Omega)\}$$

$$A_N = \Delta, \quad D(A_N) = \{u \in L_2(\Omega) \mid u \in H^1(\Omega), \ \Delta u \in L_2(\Omega), \frac{\partial u}{\partial n} = 0 \text{ in a weak sense}^{(8)}\}.$$

Remark. By $H^m(\Omega)$ we denote the space completion of the linear space $C^m(\bar{\Omega})$ under the norm induced by the inner product $(u,v)_{H^m}$ $= \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u \, \overline{D^\alpha v} \, dx$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, and by $H^m_0(\Omega)$ the closure in $H^m(\Omega)$ of $C_0^\infty(\Omega)$ [9].

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(8) Precisely in the sense that $-\int_\Omega \Delta u \bar{v} dx = \int_\Omega (\nabla u . \nabla \bar{v}) dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} \bar{v} \, ds$ $= \int_\Omega \nabla u . \nabla \bar{v} dx$, $\forall v \in H^1(\Omega)$. By $\nabla u . \nabla v$ we mean $\sum_{n=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}$ as usual, and the overbar denotes complex conjugate.

(9) For a comprehensive study of these Sobolev spaces see [5] .

In either of the above cases,

$$\text{Re } ((\lambda I-A)u,u) = \lambda\int_\Omega u\bar{u}dx + \int_\Omega |\nabla u|^2 dx \geq K|u|^2_{H^1}, \quad K>0 \text{ constant.}$$

Then, Lax - Milgram Theorem [6,p. 11] implies that $\lambda I-A:D(A) \to L_2(\Omega)$ is one to one and onto for $\lambda > 0$. Moreover, $|(\lambda I-A)u| \geq \lambda|u|$, and, hence, $\|(\lambda I-A)^{-1}\| \leq \frac{1}{\lambda}$, $\lambda>0$. Consequently, either A_D or A_N generate a contraction semigroup.

<u>Example 3.</u> Consider now the hyperbolic equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = Au(x,t), \quad A=a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i\frac{\partial}{\partial x_i} + c, \quad x\in\Omega\subset R^n, \ t>0,$$

$a_{ij} = a_{ji}$, b_i, c smooth functions, together with $u(x,t) = 0$ on $\partial\Omega$, and $u(x,0) = u_o(x)$, $\partial u(x,0)/\partial t = \dot{u}_o(x)$ for $x\in\Omega$. We assume that A is strongly elliptic, i.e., there exists $\varepsilon>0$ such that $a_{ij}(x)\eta_i\eta_j \geq \varepsilon|\eta|^2$, for all $x\in\Omega$, $\eta\in R^n$ (obviously, we are using the summation convention).

If $D(A)=\{u\in H^2(\Omega)\,|\,u=0 \text{ on } \partial\Omega\}$, and we consider the operator

$$B = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

with domain $D(B) = D(A)\times H^1(\Omega) \subset X = H^1(\Omega)\times L_2(\Omega)$, the abstract version of the problem can be written

$$\frac{d}{dt}\begin{pmatrix} u \\ \dot{u} \end{pmatrix} = B\begin{pmatrix} u \\ \dot{u} \end{pmatrix}, \quad \begin{pmatrix} u(o) \\ \dot{u}(o) \end{pmatrix} = \begin{pmatrix} u_o \\ \dot{u}_o \end{pmatrix}, \tag{7}$$

which is the usual form for the application of the semigroup method.

We need here the following result from elliptic theory (Garding's Inequality [7,8,9]) : If A is strongly elliptic, then there exist constants $\alpha>0$, $\gamma>0$ such that

$$\phi[u,u] \equiv \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \geqslant \alpha |u|^2_{H^1} - \gamma |u|^2_{L^2}, \quad \forall\ u \in H^1(\Omega).$$

This inequality implies $-(Au,u) \geqslant \alpha_1 |u|^2_{H^1} - \gamma_1 |u|^2_{L_2}$, $\alpha_1, \gamma_1 > 0$. $u \in H^2(\Omega)$, $u = 0$ on $\partial\Omega$, as is easily deduced by using Young's inequality: $2\,ab \leqslant \varepsilon\, a^2 + \frac{1}{\varepsilon} b^2$, ε arbitrary [10]. Thus, $((\lambda I - A)u,u) \geqslant \alpha_1 |u|^2_{H^1}$ for $\lambda > \gamma_1$.

It is obvious that B is densely defined in X. On the other hand, and in view of Garding's inequality, we can choose in $H^1(\Omega)$ the equivalent norm $\phi[u,u] + \gamma\ |u|^2_{L_2} \equiv |u|^2_*$ -notice that $|\phi[u,v]| \leqslant C|u|_{H^1} |v|_{H^1}$-. Then, it is easy to see that, using this norm,

$$(B\binom{u}{\dot u},\ (u,\dot u))_X = \phi[u,\dot u] + \gamma(u,\dot u)_{L_2} + (Au,\dot u)_{L_2}$$

$$\leqslant K\ (|\dot u|^2_{L_2} + |u|^2_*), \quad K \text{ constant}.$$

Let now u_1 and u_2 be the solutions of

$$\lambda^2 u_1 - Au_1 = v_1$$
$$\lambda^2 u_2 - Au_2 = v_2\ ,$$

whose existence is a consequence of Lax-Milgram theorem and regularity results -that is, if $u \in H^2(\Omega)$ and it satisfies the boundary conditions, then $|u|_{H^m} \leqslant \bar K\ (|Au|_{H^{m-2}} + |u|_{L_2})$, $m \geqslant 2$, $\bar K$ constant [7]-. Thus $u = \lambda u_1 + u_2$, $\dot u = \lambda u_2 + Au_1$ is a solution to

$$(\lambda I - B)\ \binom{u}{\dot u} = \binom{v_1}{v_2},$$

so that $\lambda I - B$ is onto for $|\lambda|$ sufficiently large. Consequently, $B \in G(X,1,\beta)$ by virtue of Corollary 2.

<u>Example 4.</u> Let $\Omega \subset R^n$ be a smooth region, and consider the equations of linear elastodynamics [11, Sec. 59] in the absence of body forces

$$\rho(x)\ \frac{\partial^2 u_i(x,t)}{\partial t^2} = \frac{\partial}{\partial x_j}\ [\,C_{ijk\ell}(x)\ \frac{\partial u_k(x,t)}{\partial x_\ell}\,]\ ,\quad x\in\Omega\ ,$$

with boundary conditions to be either $u_i = 0$ or $C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}n_j = 0$ on $\partial\Omega$, n_j unit normal to $\partial\Omega$, and where $C_{ijk\ell}(x) = C_{ji\ell k}(x) = C_{k\ell ij}(x)$ are assumed to be smooth, and $\rho(x)\geqslant \rho_o > 0$ continuous. We take the $L_2(\Omega)$ inner product weighted by ρ, and write $Au\equiv\frac{1}{\rho}\ \frac{\partial}{\partial x_j}(C_{ijk\ell}\frac{\partial u_k}{\partial x_\ell})$ with domain $D(A) =\{u\in H^2(\Omega)\,|\ u$ satisfying the boundary conditions$\}$.

Following the same procedure as in the previous example, we can write the equation of balance of linean momentum as (7), with $D(B) = D(A)\times H^1(\Omega)\subset X = H^1(\Omega)\times L_2(\Omega)$. In this case, the strong ellipticity condition reads

$$C_{ijk\ell}(x)\,\xi_i\xi_k\eta_j\eta_\ell \geqslant \varepsilon|\xi|^2|\eta|^2,\ \ \varepsilon>0,\xi,\eta\in R^n,\ x\in\bar{\Omega},$$

and Garding's inequality becomes

$$\phi[\,u,u\,]\equiv\int_\Omega C_{ijk\ell}\ \frac{\partial u_i}{\partial x_j}\ \frac{\partial u_k}{\partial x_\ell}\ dx \geqslant \alpha|u|^2_{H^1}-\gamma|u|^2_{L_2},\alpha>0,\gamma\geqslant 0,$$

for all $u_i\in H^1(\Omega)$, iff $C_{ijk\ell}$ is strongly elliptic. It is important to consider the case in which the elastic potential energy satisfies the positive definiteness condition

$$\phi\ [u,u] \geqslant K|u|^2_{H^1}\ ,\qquad K>0,\qquad \forall\ u_i\in H^1(\Omega).\qquad\qquad (8)$$

Then, the following statements hold true:

1.- The operator $B\in G(X,1,\beta)$, relative to some Hilbert norm in X, if and only if $C_{ijk\ell}$ is strongly elliptic.

2.- The operator $B\in G(X,1,0)$, relative to some norm in X, if and only if (8) is satisfied.

The proof of the "if" part of 1 is similar to that in Example 3; for the converse, we use (iii) of Corollary 2 and the possibility of choosing $\dot{u} = \alpha u$, $\alpha > 0$, $\alpha > \beta$. Result 2 is a consequence of 1 taking $\phi[u,v]$ as equivalent inner product on H^1 (10).

Now, we shall briefly consider an important class of semigroups which arise in evolution problems of parabolic type, such as Example 2; namely, these semigroups, called analytic, arise in equations of the form $\frac{\partial u}{\partial t} = Au$, where A is an elliptic differential operator satisfying the hypotheses of Garding's inequality.

Let $\{U(t)\}$, $t \geqslant 0$, be the semigroup generated by A. The semigroup $\{U(t)\}$ is said to be differentiable if $U(t)v \in D(A)$ for all $v \in X$ and $t > 0$. If $U(t)$ is differentiable, then

$$\frac{d^+}{dt} U(t)v = \lim_{h \downarrow 0} h^{-1}[U(h)U(t)v - U(t)v] = AU(t)v$$

$$= U(t/\ )AU(t/_2)v = \lim_{h \downarrow 0} U(t/_2 - h)h^{-1}[U(t/2+h)v - U(t/_2)v]$$

$$= \frac{d^-}{dt} U(t)v, \quad t \in (0,\infty), \quad v \in X,$$

and, since $U(h)v \in D(A)$, $t \geqslant h > 0$, we have $AU(t)v = U(t-h)AU(h)v$; thus, we conclude that $U(\cdot)v : (0,\infty) \to X$ is continuously differentiable, with $\frac{d}{dt} U(t)v = AU(t)v$, for all $t > 0$, $v \in X$ (11). A semigroup $\{U(t)\}$, $t \geqslant 0$, is said to be analytic if it is differentiable and there exists a constant $C > 0$ such that

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(10) For details, as well as many interesting comments, see [4]. Also, for an elastic case with thermal and memory dissipation giving rise to a system of functional-differential equations, see [4'].

(11) It can be proved [2] that $U : (0,\infty) \to B(X)$ is continuously differentiable and $\frac{d}{dt} U(t) = AU(t)$, for all $t > 0$.

$$\| AU(t) \| \leqslant Ct^{-1}, \quad \text{for all } t \in (0,1] .$$

It can be proved [12] that, in this case, U(t) can be continued analitically into the sector $S(\alpha)=\{\lambda \in C \mid |\arg \lambda|<\alpha,\ 0<\alpha<\pi/2\}$, and the continuation, $\hat{U}(z) \in B(X)$, satisfies $\hat{U}(z_1+z_2)=\hat{U}(z_1)\hat{U}(z_2)$, $\hat{U}(z)v \to v$ as $z \to 0$, $z_1, z_2, z \in S(\alpha)$, and $\hat{U}(t)=U(t)$ for all $t \in (0,\infty)$.

The next theorem gives a sufficient condition for the linear operator A to be the generator of a uniformly bounded analytic semigroup (for the general case we obviously replace A by $A+I\beta$):

<u>Theorem 2 [1,2,13]</u>. Assume A is a closed densely defined linear operator on X whose resolvent set $\varphi(A)$ contains the sector $S(\alpha+\frac{\pi}{2})$, $\alpha \in (0,\pi/2)$, and there exists a constant M such that $\|(\lambda I-A)^{-1}\| \leqslant M|\lambda|^{-1}$ for all $\lambda \in S(\alpha+\pi/2)$. Then, A is the generator of a unique analytic semigroup, U(t), satisfying $\| U(t) \| \leqslant C$ and $\| AU(t) \| \leqslant Ct^{-1}$ for all $t \in (0,\infty)$, where C is a constant.

If, in the above theorem, $\|(\lambda I-A)^{-1}\| \leqslant M(|\lambda|+q)^{-1}$ for some $q>0$, then there is a $\delta > 0$ such that

$$\| U(t) \| \leqslant C^{-\delta t} \quad \| AU(t) \| \leqslant t^{-1}C^{-\delta t} \quad \text{for all } t \in (0,\infty);$$

namely, the analytic semigroup is exponentially decaying [2].

An interesting regularity result of immediate proof is the following: Let A be a given linear operator satisfying the hypotheses of Theorem 2, then $U(t)v \in D(A^n)$, $n=1,2,\ldots$, for any $v \in X$, $t \in (0,\infty)$, and $\| A^n U(t) \| \leqslant K\, t^{-n}$, $t>0$, where K is a constant depending only on A,n.

Let now $f:[0,T] \to X$ be a continuous function, $A \in G(X,M,\beta)$ and consider the Cauchy Problem

$$\dot{u}(t) = Au(t) + f(t), \quad t>0, \quad u(0) = v. \qquad (9)$$

By assuming that this <u>nonhomogeneous initial value problem</u> admits a solution $u(t) \in D(A)$ for all $t \in [0,T]$, and if $\{U(t)\}$ is the semigroup whose generator is A, we have

$$\frac{d[U(t-s)u(s)]}{ds} = U(t-s)\frac{du(s)}{ds} - \frac{dU(t-s)}{ds}u(s) = U(t-s)f(s), \quad s \in (0,t),$$

and, hence,

$$u(t) = U(t)v + \int_0^t U(t-s)f(s)ds \qquad t \in [0,T]. \qquad (10)$$

The continuous function defined by (10) is called the <u>mild solution</u> of (9), since it is possible for such a solution not to be differentiable (12). The following theorem gives the main result on the existence of solutions to (9):

<u>Theorem 3 [2].</u> The mild solution is the unique solution to (9) over the time interval $[0,T]$ if (i) $f(t) \in D(A)$ for $0 \leqslant t \leqslant T$, and $Af \in C([0,T];X)$, or if (ii) $f \in C^1([0,T];X)$. In both cases, if the semigroup $\{U(t)\}$ is also differentiable, then the mild solution is also a solution for each $v \in X$.

If $f(t) = B(t,u(t))$, where $B:[0,T] \times D \to X$, $D \subset X$, is continuous, we have the <u>semilinear</u> initial value problem

$$\dot{u}(t) = Au(t) + B(t,u(t)) \text{ for } t>0, \; u(0) = v \in D. \qquad (11)$$

The fundamental technique to deal with (11) is the study of existence of mild solutions, followed by the analysis of differentiability properties of such solutions. All these results will be local in time and, as a last step, one can consider global existence of solutions(13). However, we shall mention a much more general case later, and here we limit ourselves to state two interesting perturbation results for the

(12) For instance, if there is $w \in X$ such that $U(t)w \notin D(A)$ for any $t \geqslant 0$, and $f(t)=U(t)w$ for all $t \geqslant 0$.

(13) See, for instance [2] for a comprehensive study.

autonomus situation in which B is time independent.

Theorem 4 [14]. If $A\in G(X,M,\beta)$ and $B\in B(X)$. Then $A+B\in G(X,M,\beta+M\|B\|)$.

In general, it is difficult to have a similar result when the perturbation B is an unbounded operator.

Theorem 5 [15]. Assume that $A\in G(X,M,\beta)$, and let $B:D(B)\subset X\to X$, with $D(B)$ convex and open, be such that $B(0)=0$ and $|B(v)-B(v_1)| \leqslant a|v-v_1|$, $a>0$, for all $v,v_1 \in D(B)$. Then, if $u(0)\in D(A)\cap D(B)$, there exists a unique mild solution for $t\in[0,T]$, provided that $T>0$ is suitably small. Moreover, if B is Frêchet differentiable [1,24] at any $v\in D(B)$, and its Frêchet derivative B_v satisfies

(i) $|B_v w|\leqslant b|w|$, $b>0$, for all $v\in D(B)$ and all $w\in X$,

(ii) $|B_v w-B_{v_1} w| \to 0$ as $|v-v_1|\to 0$, for all $w\in X$ and for all $v,v_1 \in D(B)$,

then we have a solution.

It is easy to prove that if $|u(t)|\leqslant \delta$, $t\in[0,\bar{T}]$, where δ is a constant and $\bar{T}$ an arbitrarily fixed value of time, then the solution $u(t)$ is defined over the whole interval $[0,\bar{T}]$.

We end this part concerning linear equations of evolution by considering the non autonomous problem

$$\dot{u}(t) = A(t)u(t) + f(t) \text{ for } t\in[0,T] \text{ and } u(0) = v, \qquad (12)$$

where $\{A(t)\}$, $0\leqslant t\leqslant T$, is a family of operators belonging to the set of all generators of strongly continuous semigroups on X. In early studies two non mutually exclusive cases have been considered: the hyperbolic case in which $A(t)$ is, for each t, the infinitesimal

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(14) The proof of Theorem 4, as well as many others important pertubation and approximation results, can be found in [3] and [14].

generator of a contraction semigroup [16,17], and the parabolic case in which $A(t)$ generates, for each t, an analytic semigroup [1,13,18]. In both cases, a fundamental assumption is the time independence of the domain of $A(t)$. A more recent result, eliminating the restriction on $D[A(t)]$ just mentioned, is the following:

Theorem 6 [19]. Let the family $\{A(t)\}$, $0 \leq t \leq T$, be stable with stability index M,β; namely, there are numbers M,β such that

$$\left\| \prod_{m=1}^{n} (\lambda I - A(t_m))^{-1} \right\| \leq M(\lambda-\beta)^{-n}, \qquad \lambda > \beta,$$

for every finite set $0 \leq t_1 \leq \ldots \leq t_n \leq T$, $n=1,2,\ldots$, where the operator product is time ordered [15]. Assume that there is a Banach space Y, continuously and densely embedded in X, and an isomorphism S of Y onto X such that

$$SA(t)S^{-1} = A(t) + B(t), \quad B(t) \in B(X), \quad t \in [0,T],$$

where $B(\cdot)v: [0,T] \to X$ is strongly measurable for each $v \in X$ [1,17], and where $B(t)$ is upper integrable on $[0,T]$. Furthermore, assume that the restriction of $A(t)$ to Y belongs to $B(Y,X)$ and the map $A:[0,T] \to B(Y,X)$ is continuous in norm. Then, there exists a unique evolution operator $U(t,s)$, defined on $\Delta: 0 \leq s \leq t \leq T$, with the following properties: (i) $U:\Delta \to B(X)$ is strongly continuous; (ii) $U(s,s) = I$, and $U(t,\tau)U(\tau,s) = U(t,s)$; (iii) $U(t,s)Y \subset Y$, and $U: \Delta \to B(Y)$ is strongly continuous; (iv) the derivatives $\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s)$, $\frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s)$, exist in the strong sense in $B(Y,X)$ and are strongly continuous on Δ to $B(Y,X)$. Moreover, if u is the mild solution

$$u(t) = U(t,0)v + \int_0^t U(t,s)\, f(s)\, ds$$

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(15) Note that $\{A(t)\}$ is trivially stable, with stability index $1,\beta$ if $A(t) \in G(X,1,\beta)$.

of (12), $v \in Y$ and $f \in C([0,T];X) \cap L_1([0,T];Y)$, then $u \in C([0,T];Y) \cap C^1([0,T];X)$ and u satisfies (12).

Remark 2. Obviously, the unique solution of the homogeneans equation $\dot{u}(t)=A(t)u(t)$, with initial value $u(s)=\bar{v}$, is given by $u(t)=U(t,s)\bar{v}$, and, in the particular case $A(t)=A=$ constant, $U(t,s)$ becomes the semigroup $U(t-s)$ generated by A. Also, the mild solution $u \in C([0,T];X)$ if $v \in X$ and $f \in L_1([0,T];X)$, and $u \in C([0,T];Y)$ if $v \in Y$, $f \in L_1([0,T];Y)$ hold true [19].

2. Nonlinear semigroups.

In this section we consider non-linear evolution equations. H will be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|.|$.

Definition. A multivalued operator is a mapping $A:H \to P(H)$, where $P(H)$ is the collection of subsets of H. The domain of A is the set $D(A)=\{v \in H \mid Av \neq \emptyset,\ \emptyset$ the empty set$\}$, and the range of A the set $R(A)=\bigcup_{v \in H} Av$. We shall identify A with its graph, and write $A \subset H\times H$.

If $A \subset H\times H$ and $B \subset H\times H$, with α and β real, then

$$\alpha A+\beta B=\{[v,\alpha u+\beta w] \mid [v,u] \in A,\ [v,w] \in B\},$$

with domain $D(A) \cap D(B)$. The operator A^{-1} is defined as $A^{-1}=\{[u,v] \mid [v,u] \in A\}$ which always exists as a multivalued map. Obviously, $I=\{[v,v] \mid v \in H\}$ is the identity operator, and $D(A^{-1}) = R(A)$.

Remark 3. The reason for introducing multivalued maps is the one to one correspondence which can be established between non-linear contraction semigroups and a certain class of multivalued operators, generalizing to nonlinear problems the Hille-Yosida Theory for linear semigroups. On the other hand, certain unilateral problems [20] can be very naturally formulated in terms of multivalued operators.

Definition. The operator $A \subset H \times H$ is called monotone if

$$(u_1-u_2, v_1-v_2) \geqslant 0 \text{ for all } [v_1,u_1],[v_2,u_2] \in A,$$

and maximal monotone if there is no monotone graph different from A and containing A, namely,

$$(u_1-u_2, v_1-v_2) \geqslant 0 \text{ for all } [v_2,u_2] \in A, \text{ implies } [v_1,u_1] \in A.$$

The resolvent of A, denoted now by J_λ, is the operator

$$J_\lambda \equiv (I+\lambda A)^{-1} = \{[v+\lambda u, v] \mid [v,u] \in A\}.$$

Note that, for all $\lambda>0$, each element in $D(J_\lambda)=R(I+\lambda A)$ admits a unique representation as $v+\lambda u, [v,u] \in A$. For, suppose $v_1+\lambda u_1 = v_2+\lambda u_2$; then, $(v_1-v_2+\lambda(u_1-u_2),\ v_1-v_2) = |v_1-v_2|^2+\lambda(u_1-u_2, v_1-v_2) = 0$, and, hence, $|v_1-v_2|^2 \leqslant 0$, which implies $v_1=v_2$, $u_1=u_2$. Moreover, $|(v_1+\lambda u_1)-(v_2+\lambda u_2)|^2 \geqslant |v_1-v_2|^2$. Thus J_λ is a single valued contraction.

The following Proposition gives a fundamental characterization of maximal monotone operators:

Proposition [21]. Let A be an operator on H. The following three statements are equivalent: (a) A is maximal monotone; (b) A is monotone and $R(I+A)=H$; (c) for all $\lambda>0$, J_λ is a contraction defined on H [16] -clearly, for every $w \in H$ we can solve the equation $(I+A)v=w$-.

It is easy to see that, if A is maximal monotone, then the set Av is closed and convex. Let $A°v$ be the unique element of minimum norm in Av [17]. The single valued operator defined by $[v, A°v]$ is called the

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[16] That is, for A maximal monotone, $D(J_\lambda)$ is a "nice" domain, in contrast to the usual case where $D(J_\lambda)$ can be "wild".

[17] Recall that a non-empty closed convex subset of a Hilbert space has a unique element of minimal norm (see, for instance [22], p. 15).

minimal section.

Remark 4. Suppose A is maximal monotone. It is straightforward to verify $J_\lambda w = J_\mu\{(\mu/\lambda)w + (1-\mu/\lambda)J_\lambda w\}$ for all $w \in H$ and for all $\lambda, \mu > 0$, equality which reduces to the usual resolvent identity in the linear case. Also, the closure $\overline{D(A)}$ is convex and $\lim_{\lambda\to 0} J_\lambda w = \mathrm{Proj}_{\overline{D(A)}} w$, i.e., J_λ approximates the identity [21].

The single-valued operator $A_\lambda \equiv \lambda^{-1}(I - J_\lambda)$ is called Yosida approximation [18] and posseses the following properties [21]: (i) A_λ is everywhere defined, maximal monotone, and Lipschitz with constant $1/\lambda$; (ii) $(A_\lambda)_\mu = A_{\lambda+\mu}$ for all $\lambda, \mu > 0$; (iii) when $\lambda \downarrow 0$, $|A_\lambda w| \uparrow |A^\circ w|$, for $w \in D(A)$ with $|A_\lambda w - A^\circ w|^2 \leqslant |A^\circ w|^2 - |A_\lambda w|^2$; (iv) if $w \notin D(A)$, $|A_\lambda w| \uparrow \infty$ as $\lambda \downarrow 0$.

To end this review on results for maximal monotone operators, we state the following Perturbation Theorems [21]:

1) Let A be a maximal monotone operator, and let B be a monotone, Lipschitz and everywhere defined operator. Then, A+B is maximal monotone.

2) Let A and B be maximal monotone operators. The element $w \in R(I+A+B)$ if and only if $|B_\lambda v_\lambda| \leqslant$ constant, for $\lambda > 0$ in $w = v_\lambda + u_\lambda + B_\lambda v_\lambda$, $u_\lambda \in Av_\lambda$ (note that, by the previous theorem, $R(I+A+B_\lambda) = H$). Moreover, in this case, $v_\lambda \to v$ as $\lambda \to 0$, where v is a solution of $w = v + u + \bar{u}$, $u \in Av$, $\bar{u} \in Bv$.

3) If A and B are maximal monotone operators with $D(A) \subset D(B)$, and if there are $k < 1$ and $g: R \to R$ such that $|B^\circ v| \leqslant k|A^\circ v| + g(|v|)$ for all $v \in D(A)$, then A+B is maximal monotone.

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(18) Note that AJ_λ is multivalued, and we have the obvious inclusion $A_\lambda w \in AJ_\lambda w$ for all $w \in H$.

4) If A and B are maximal monotone operators such that $(\mathrm{Int}D(A))\cap D(B)\neq\emptyset$, then A+B is maximal monotone and $\overline{D(A)\cap D(B)} = \overline{D(A)}\cap\overline{D(B)}$.

Now, we shall consider the initial value problem

$$0\in \dot{u}+Au \quad (\dot{u}+Au=0 \text{ if } A \text{ single-valued}),\ u(0)=u_o. \tag{13}$$

A solution of this initial value problem is a function u(t) satisfying (i) $u\in C([0,T];H)$, $u(0)=u_o$; (ii) u(t) is Lipschitz on [0,T] [19]; (iii) the equation above is satisfied almost everywhere on [0,T].

Theorem (Uniqueness). If A is monotone the initial value problem (13) has at most a solution.

For, if $0\in\dot{u} + Au$, $u(0)=u_o$, and $0\in\dot{v} +Av, v(0)=v_o$, substracting we obtain $\dot{u}-\dot{v}+w-\bar{w}=0$, where $w\in Au$, $\bar{w}\in Av$. Performing the scalar product with u-v, integrating, and recalling that A is monotone, we get $|u(t)-v(t)| \leqslant |u_o-v_o|$.

Hence, the mapping $U(t):u_o \to u(t)$ will satisfy

$$|U(t)u_o-U(t)v_o| \leqslant |u_o-v_o|,$$

and, if we have existence for $t\geqslant 0$, $\{U(t)\}$ will be a continuous semigroup of contractions which can be extended by continuity to $\overline{D(A)}$, and -A will be its generator.

Theorem (Existence). Let A be maximal monotone. Then, for every $u_o\in D(A)$ the initial value problem (13) has a solution $u(t)\in D(A)$ for all $t>0$.

Outline of Proof [21]. As in the linear case, we can consider the equation

$$\dot{u}_\lambda(t)+A_\lambda u_\lambda(t) = 0, \quad u_\lambda(0)=u_o, \tag{14}$$

which has a solution of class C^1 on $[0,\infty)$, since A_λ is Lipschitz.

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(19) In a reflexive Banach space the Lipschitz condition implies differentiability almost everywhere and the function is the integral of the derivative ([21], Appendix).

Then, we prove at the first place that u_λ is a Cauchy sequence in $C([0,T];H)$ as $\lambda \to 0$. To see this, one follows the same line as in the uniqueness proof, and using $A_\lambda u_\lambda \in AJ_\lambda u_\lambda$, the identity $u_\lambda - u_\mu = \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu + J_\lambda u_\lambda - J_\mu u_\mu$, and the monotonicity of A with $J_\lambda u_\lambda, J_\mu u_\mu$, the estimate

$$|u_\lambda(t) - u_\mu(t)| \leqslant 2^{1/2} \, |A^o u_o| \, (\lambda+\mu)^{1/2} \, T^{1/2} \qquad (15)$$

is obtained, where we have taken into account that $|A_\lambda u_\lambda(t)| \leqslant |A_\lambda u_\lambda(o)| \leqslant |A^o u_o|$ for $u_o \in D(A)$. Hence, $u_\lambda(t) \to u(t)$ uniformly on $[0,T]$ as $\lambda \to 0$, for all $T < \infty$. Thus, $u \in C([0,T];H)$, $u(o) = u_o$ and $u(t)$ is Lipschitz with constant $|A^o u_o|$.

Secondly, $u(t) \in D(A)$ for all $t > 0$. For, $|u_\lambda(t) - J_\lambda u_\lambda(t)| \leqslant |A^o u_o| \lambda$ and $J_\lambda u_\lambda(t) \to u(t)$ uniformly on $[0,T]$ as $\lambda \to 0$. Moreover, by the Eberlein-Shmulyan Theorem [17] there exists a sequence $\lambda_n \to 0$ such that $A_{\lambda_n} u_{\lambda_n} \in A \, J_{\lambda_n} u_{\lambda_n}$ converges weakly to v for t fixed. Then, $(\bar{w} - A_{\lambda_n} u_{\lambda_n}, w - J_{\lambda_n} u_{\lambda_n}) \geqslant 0$ for all $[w,\bar{w}] \in A$, and hence $(\bar{w} - v, w - u) \geqslant 0$, that is, $u(t) \in D(A)$ and $v(t) \in Au(t)$.

Finally, in obtaining estimate (15), we had

$$\int_o^t (A_\lambda u_\lambda - A_\mu u_\mu, \lambda A_\lambda u_\lambda - \mu A_\mu u_\mu) ds \leqslant 0,$$

and then $A_\lambda u_\lambda(t) \to v(t)$ in $L_2([0,T];H)$ (20). Therefore, we have $A_{\lambda_n} u_{\lambda_n}(t) \to v(t)$ almost everyhere as $\lambda_n \to 0$, so that writing the differential equation (14) in integral form, and passing to the limit, we get

$$u(t+h) - u(t) = -\int_t^{t+h} v(s) \, ds,$$

(20) Here, we apply that if X is a real Hilbert space, and $Z_n \in X$, $\lambda_n \in R^+$, such that $2(Z_n - Z_m, \lambda_n Z_n - \lambda_m Z_m) = (\lambda_n + \lambda_m)|Z_n - Z_m|^2 + (\lambda_n - \lambda_m)(|Z_n|^2 - |Z_m|^2) \leqslant 0$, then if λ_n is monotone decreasing and $|Z_n| \leqslant$ constant, the sequence Z_n converges.

where we have applied the Lebesgue dominated convergence theorem[23]. Dividing above by h and letting $h \to 0$, we have

$$\frac{du(t)}{dt} = -v(t) \in Au(t)$$

at any t where we have convergence.

Theorem (Properties of the solution) [21]. Let u(t) be the solution of the initial value problem (13). Then, (i) $A^{o}u(t)$ is continuous from the right, and $|A^{o}u(t)|$ decreases with time; (ii) u(t) is everywhere differentiable from the right and

$$\frac{d^{+}u(t)}{dt} + A^{o}u(t) = 0 \quad , \qquad t \geqslant 0.$$

Remark 5. If A is single-valued $Au(.): [0,\infty) \to H$ is weakly continuous and u is weakly differentiable on $(0,\infty)$.

Remark 6. It can also be proved [21] that, given a contraction semigroup defined on a closed and convex set D, there is a unique maximal monotone operator A with $\overline{D(A)}=D$, and -A is the generator of the semigroup.

Example 5. Let $f:H \to (-\infty,\infty]$, $f \not\equiv \infty$, be a lower semicontinuous and convex function on H; namely, for any sequence $\{u_n\}$ with $u_n \to u$, $f(u) \leqslant \liminf f(u_n)$, and $f(tu+(1-t)v) \leqslant tf(u)+(1-t)f(v)$ for $t \in [0,1]$. The set $D(f)=\{u \in H | f(u)<\infty\}$ is convex. An element $u \in H$ is called a subgradient of f at v if f(v) is finite and

$$f(w)-f(v) \geqslant (w-v,u) \qquad \text{for all } w \in H.$$

The set of these subgradients is called the subdifferential, $\partial f(v)$, of f at the element v [(21)]. The subdifferential, ∂f, is a multivalued

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(21) When f is weakly differentiable at v with u as Gateaux differential at this point, the subdifferential reduces to the singleton $\{u\}$ [24].

maximal monotone operator. For, if $u_1 \in \partial f(v_1)$ and $u_2 \in \partial f(v_2)$, we have

$$f(v_2)-f(v_1) \geqslant (v_2-v_1, u_1), \quad f(v_1)-f(v_2) \geqslant (v_1-v_2, u_2);$$

thus, $(u_1-u_2, v_1-v_2) \geqslant 0$ for all $u_1 \in \partial f(v_1)$, $u_2 \in \partial f(v_2)$ and hence ∂f is monotone. To prove the maximality we use the following result [21]: For fixed $u \in H$, the function $f(v)+\frac{\lambda}{2}|v-u|^2$ is convex, and has a minimum v_o, if and only if $\lambda(u-v_o) \in \partial f(v_o)$; then, since $f(v)+\frac{1}{2}|v-u|^2$ is a convex lower semicontinuous function which goes to ∞ as $|v| \to \infty$, it will reach its minimum in v_o, and, hence, $u-v_o \in \partial f(v_o)$ for all $u \in H$.

It is straightforward to verify that the operator A_N of Example 2 is the subdifferential of the convex lower semicontinuous function $f(u)=\frac{1}{2}\int_\Omega |\nabla u|^2 dx$.

Certain maximal monotone operators generate semigroups for which we have a regularizing situation similar to the case of analytic semigroups. A relevant theorem is the following one:

<u>Theorem [21].</u> Let A be the subdifferential of a convex lower semicontinuous function, and U(t) the semigroup generated by -A. Then, for every $u \in \overline{D(A)}$, $U(t)u \in D(A)$ for $t>0$, and, moreover,

$$|A^o U(t)u| \leqslant t^{-1}|u-U(t)u| \qquad t>o,$$

$$|A^o U(t)u| \leqslant |A^o v| + t^{-1}|v-u| \qquad t>o,\ v \in D(A).$$

<u>Remark 7.</u> Let f be an element of $L_1([0,T];X)$. The function $u \in C([0,T];X)$ is called a <u>weak solution</u> of $f \in \frac{du}{dt} + Au$, $u(o) = v$, if there are sequences $\{f_n\} \subset L_1([o,T];X)$, $\{u_n\} \subset C([o,T];X)$, such that $f_n \to f$ in L_1 and $u_n \to u$ uniformly on $[0,T]$, u_n being absolutely continuous and satisfying

$$f_n \in \frac{du_n}{dt} + Au_n \qquad \text{a.e.}, \ u_n(o)=v_n;$$

namely, u_n is a solution (strong) of the initial value problem above.

Remark 8. In Banach spaces monotonicity is replaced by a condition called accretiveness: $A \subset X \times X$ is accretive if

$$|(u_1+\lambda v_1)-(u_2+\lambda v_2)| \geqslant |u_1-u_2| \text{ for all } \lambda>0 \text{ and } [u_i,v_i] \in A, i=1,2.$$

One can define

$$\langle u,v\rangle_+ = \lim_{\lambda\downarrow 0} \lambda^{-1}(|u+\lambda v|-|u|),\quad \langle u,v\rangle_- = \lim_{\lambda\uparrow 0} \lambda^{-1}(|u+\lambda v|-|u|),$$

$\lambda \to \lambda^{-1}(|u+\lambda v|-|u|)$ being a convex and nondecreasing function. Then, A is accretive iff $\langle u_1-u_2, v_1-v_2\rangle_+ \geqslant 0$ for all $[u_i,v_i] \in A$, $i=1,2$.

Relevant results concerning solutions of (13) in a Banach space X are the following:
Let A be an accretive operator satisfying $\overline{D(A)} \subset R(I+\lambda A)$ for $\lambda>0$ sufficiently small. Then, for every $v \in \overline{D(A)}$, U(t) defined by $\lim_{n\to\infty} (I+n^{-1}tA)^{-n} v$ has the properties: (i) $U(t) : \overline{D(A)} \to \overline{D(A)}$; (ii) $U(t+s)=U(t)\,U(s)$; (iii) $U(t)v \to U(s)v$ as $t \to s$; (iv) $|U(t)v-U(t)w| \leqslant |v-w|$; (v) if $v \in D(A)$; then $U(\cdot)v$ is Lipschitz. If the initial value problem posseses a solution (strong) for $u_o \in D(A)$, this solution is given by $U(t)u_o$. (The reader is referred to [30,28]; see also [25,26,27,29,31,32,2] for the theory as well as examples). However, it may happen that A satisfies suitable conditions, but for no $u_o \in D(A)$ does the initial value problem (13) have a solution (e.g. [30]). These results can be extended to semigroups of type β, i.e., those for which $|U(t)v-U(t)w| \leqslant e^{\beta t}|v-w|$ for $t \geqslant 0$, $v,w \in \overline{D(A)}$.

Although the most extensively studied nonlinear evolution equations are those giving rise to nonlinear contraction semigroups, we find in many applications hyperbolic problems whose solution behaves very differently. Generally, we do not have in this case smooth solutions for all time due to the presence of shocks, and there are not even theorems ensuring the existence of unique global weak or strong solutions for interesting classes of equations. In this context we briefly consider an abstract quasilinear evolution equation which includes several problems that appear in mathematical physics:

$$\dot{u} + A(t,u)u = f(t,u), \quad t\in[0,T], \; u(0) = u_o, \qquad (16)$$

where u takes values in a Banach space X, the quasilinear part A(t,v) is a generally unbounded linear operator depending on t and $v\in X$, and the semilinear part f(t,v) depends on t and $v\in X$ [(22)]. It is assumed that -A(t,v) is the generator of a strongly continuous semigroup. The technique to deal with (16) is based on results we have mentioned at the end of Part 1 for the linear nonhomogeneous and non autonomons problem (12); that is, for certain functions $v(t)\in X$ the linear equation

$$\dot{u} + A(t,v(t))u=f(t,v(t)), \quad u(0) = u_o,$$

is considered and, if this equation has a solution u(t), one seeks, by means of the contraction mapping theorem [33], a fixed point of the mapping $\Psi v=u$, this fixed point being a solution of (16). Following this technique (outlined in [34]), theorems on existence, uniqueness, and continuous dependence on the data, as well as applications to important examples, are obtained in [35]. However, these results are not strong enough for certain important applications such as nonlinear elastodynamics, and a suitable generalization can be found in [36].

3. Asymptotic behaviour.

It is important in applications to know how fast solutions decay to an equilibrium solution. Consequently, we end this survey by giving some theorems concerning the asymptotic behaviour of linear and nonlinear semigroups.

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(22) The splitting of a given nonlinear term into quasi-linear and semilinear parts is not necessarily unique.

Linear Case. Without any restrictions on either the generator or the semigroup, is is impossible to obtain results similar to the finite dimensional case, where it is known, e.g. [37], that if the real part of the spectrum [23] of the nxn constant matrix A, $\mathrm{Re}\,\sigma(A)$, is such that $\mathrm{Re}\,\sigma(A) \leq -\gamma < 0$, then $|u(t)| \leq Me^{-\alpha t}u(0)$, for $t \geq 0$ and some $\alpha > 0$. In the setting of semigroups of bounded linear operators on a Banach space X, we have:

1) A sufficient condition for $\mathrm{Sup}\ \mathrm{Re}\,\sigma(A) = \beta, \|U(t)\| \leq Me^{\beta t}$, is that $\sigma(U(1)) \subset \{\exp\lambda \mid \lambda \in \sigma(A)\} \cup \{0\}$. Hence, if $\mathrm{Re}\,\sigma(A) \leq -\gamma < 0$, we obtain $\|U(t)\| \leq Me^{-\alpha t}$ for $t \geq 0$, $0 < \alpha < \gamma$, [38].

2) If the generator A of the (C_o) semigroup $\{U(t)\}$ satisfies $\mathrm{Re}\,\sigma(A) \leq -\gamma < 0$, and the resolvent $R(A;\lambda)$ is such that, for $\mathrm{Re}\lambda > -\gamma$, $\|R(A;\lambda)\| \sim O(|\lambda|^N)$ as $|\mathrm{Im}\lambda| \to \infty$, N some integer, then: i) if $N \geq -2$, there exists $\alpha > 0$, $0 < \alpha < \gamma$ so that $|A^{-(N+2)}U(t)v| < M(\alpha)e^{-\alpha t}|v|$ for all $v \in X$; ii) if $N \geq -1$, there exists $\alpha > 0$, $0 < \alpha < \gamma$, so that

$$\int_0^\infty |e^{\alpha t} v^*(A^{-(N+1)}U(t)v)|^2\, dt \leq M(\alpha, v, v^*), \quad M > 0, \text{ for all } v \in X, v^* \in X^*,$$

where X* denotes the dual of X, [39]. It can be seen that if U(t) is differentiable, the above condition on the resolvent is satisfied with N=1.

Nonlinear Case. To begin with, we recall a few definitions from classical topological dynamics. Let X be a real Banach space, and $D \subset X$ closed; then, for $v \in D$ the set $\gamma(v) = \{u \in X \mid u(t) = U(t)v,\ t \in R^+\}$ is the orbit through v, where U(t) is a (nonlinear) continuous semigroup of contractions on D, and $\omega(v) = \{w \in D \mid w = \lim_{n\to\infty} U(t_n)v, t_n \to \infty$ as $n \to \infty\}$ is the ω-limit set of v.

The structure of ω-limit sets characterizes the asymptotic

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(23) Recall that the spectrum of an operator is the set of complex numbers not in the resolvent set.

behaviour of solutions to $0\in\dot{u}(t) + Au(t)$. An important theorem, as far as applications is concerned, is the following [40]: Under conditions of Remark 8 (i.e., acretiveness of A, and $D=\overline{D(A)}\subset R(I+\lambda A)$ for sufficiently small λ), assume that $0\in R(A)$ and $(I+\lambda A)^{-1}$ is compact for some $\lambda>0$; then, $\gamma(v)$ is precompact for any $v\in D$.

It is easy to show [41] that, when $\gamma(v)$ is precompact, $\omega(v)$ is nonempty, compact, connected, and $U(t)v \to \omega(v)$ as $t\to\infty$. In this case, theorems 1 and 2 of [40] imply:

i) $\omega(v)$ is minimal under U, i.e., $\omega(v)=\overline{\gamma(w)}$ for any $w\in\omega(v)$; (ii) $\omega(v)$ is strongly invariant under U, i.e., for each $t\in R^+$, $U(t)$ is a homeomorphism of $\omega(v)$ onto $\omega(v)$ so that the restriction of U on $\omega(v)$ can be extended as a group on $\omega(v)$; (iii) if $U(t)v_o=v_o$ for $t\geqslant 0$, $\omega(v)$ lies on a sphere centered at v_o with radius $r\leqslant|v-v_o|$; (iv) $\omega(v)$ is equi-almost periodic under U, i.e., if beyond of being strongly invariant, for each $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that every interval of R of length $\delta(\varepsilon)$ contains a point t with the property $|U(t)w-w|<\varepsilon$ for all $w\in\omega(v)$; (v) if X is strictly convex and D convex, there is an affine group $\hat{U}$ on the closed subspace spanned by $\omega(v)$ with $U=\hat{U}$ on $\overline{co}\omega(v)$, and the fixed point, v_o, of U is given by $v_o=\lim_{t\to\infty}\int_o^t U(s)wds$ for any $w\in\overline{co}\,\omega(v)$, $\hat{U}$ being linear when $v_o=0$ (24) (in particular, if X is a Hilbert space, $\hat{U}$ is an affine group of isometries).

Various applications of these results to contraction semigroups in Hilbert spaces can be found in [43]. For the case in which A is a subdifferential we refer to [44].

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(24) For definitions of strictly convex, convex hull (denoted by co), and affine mapping see, for instance, [42].

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