Stochastica, Vol. 2. 1977

A CARACTERIZATION OF COMPLETE ATOMIC BOOLEAN ALGEBRA

by

F. Esteva

In this note we give a caracterization of complete atomic boolean algebra by means of complete atomic lattice. We find that unicity of the representation of the maximum as union of atoms and Λ-infinite distributivity law are necesary and sufficient conditions for the lattice to be a complete atomic boolean algebra.

We denote by L the lattice (L,Λ,V) with maximum u and minimum o. Let A be the atoms of L and, for every $x \in L$, I_x the set $I_x = \{a \in A \,/\, a < x\}$.

DEFINITION.- L is atomic if, for every $x \in L$, $x = \bigvee_{a \in I_x} a$.

PROPOSITION.- If L is atomic the two following conditions are equivalent :

a) For every $P \subset L$ such that $\bigvee_{x \in P} x$ exists, $\bigcup_{x \in P} I_x = I_{\bigvee_{x \in P} x}$

b) If $P \subset A$ and $\bigvee_{a \in P} a = u$, then $P = A$.

Proof.-a) implies b) for if there exists $P \subset A$, $P \neq A$ such that $\bigvee_{x \in P} x = u$, then $\bigcup_{a \in P} I_a = P \neq A = I_{\bigvee_{a \in P} a}$.

b) implies a) for if there exists $P \subset L$ such that $\bigvee_{x \in P} x = z$ exists and $\bigcup_{x \in P} I_x \quad I_{Vx}$, then $z = \bigvee_{x \in P} (\bigvee_{a \in I_x} a) = {}$ $= \bigvee_{a \in \cup I_x} a$, therefore $H = \bigcup_{x \in P} I_x \subsetneq I_z$ and $z = \bigvee_{a \in \cup I_x} a$. So $z = \bigvee_{a \in I_z} a = \bigvee_{a \in H} a$ with $H \subsetneq I_z$; so that $u = \bigvee_{a \in A} a = (\bigvee_{a \in I_z} a) V$

$\vee (\bigvee_{a\in A-I_z} a) = (\bigvee_{a\in H} a) \vee (\bigvee_{a\in A-I_z} a) = \bigvee_{a\in H \cup (A-I_z)} a$, contradicting b).

COROLARY.-If L is not distributive and it is atomic, then exist, at least, $x\in L$ and $H \subsetneq I_x$ such that $x = \bigvee_{a\in H} a = \bigvee_{a\in I_x} a$.

Proof.-If for $x\in L$ and for $H\subset I_x$, $x = \bigvee_{a\in H} a$ implies $H = I_x$, by virtue of prop.1 and knowing $I_{\wedge x} = \bigcap I_x$ for every P such that $\bigwedge_{x\in P} x$ exists, we have that the mapping $f:L \longrightarrow (P(A), \cap, \cup)$, $f(x) = I_x$ is a morfism one to one ; so that L can be identified with a sublattice of P(A), and therefore is distributive, a contradiction.

THEOREM.- Supose L is atomic and complete, the three following conditions are equivalent :

a) L is $\wedge$-infinite distributive.

b) If $P \subsetneq A$ and $\bigvee_{a\in P} a = u$, then $P = A$.

c) L is a complete atomic boolean algebra.

Proof.- a) implies b) for if L is $\wedge$-infinite distributive and there exists $P\subset A$ so that $\bigvee_{a\in P} a = u$, then exists $b\in A-P$ satisfaying $b=b \wedge (\bigvee_{a\in H \cup \bar{I}_x} a) = \bigvee_{a\in H \cup \bar{I}_x} (b \wedge a) = o$, contradiction.

b) implies c) for if b) is true, the mapping f, $f:L \longrightarrow (P(A), \cap, \cup)$, $f(x) = I_x$ is a isomorfism onto, so that L is a complete atomic boolean algebra.

c) implies a). Obvious

It is interesting to remark that the condition a) is not equivalent to "L is $\vee$-infinite distributive", as the following example shows:

Let L be the set of non negative integers having a prime decomposition with exponents one, together with zero. L with the operation L.C.M.(V) and G.C.D.(Λ), is a complete atomic V-infinite distributive lattice, but it is not a complete boolean algebra, for if P is the set of primes in L,

$$\left(\bigvee_{a\in P-\{7\}} a\right)\Lambda 7 = o\Lambda 7 \neq 1 = \bigvee_{a\in P-\{7\}}(a\Lambda 7),$$

so L is not Λ-infinite distributive.

BIBLIOGRAPHY

[1].G.BIRKOFF."Lattice theory".American Mathematical Society.Col.loquium Publication.Vol.XXV.New York,1968.

[2].G.SASZ. "Théorie des treillis".Dunod.Paris ,1971.

[3].F.ESTEVA."Negaciones en retículos completos".Stochastica n°1.Barcelona,1976.