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INTERPOLATION AND FORECASTING IN POISSON'S PROCESSES

by

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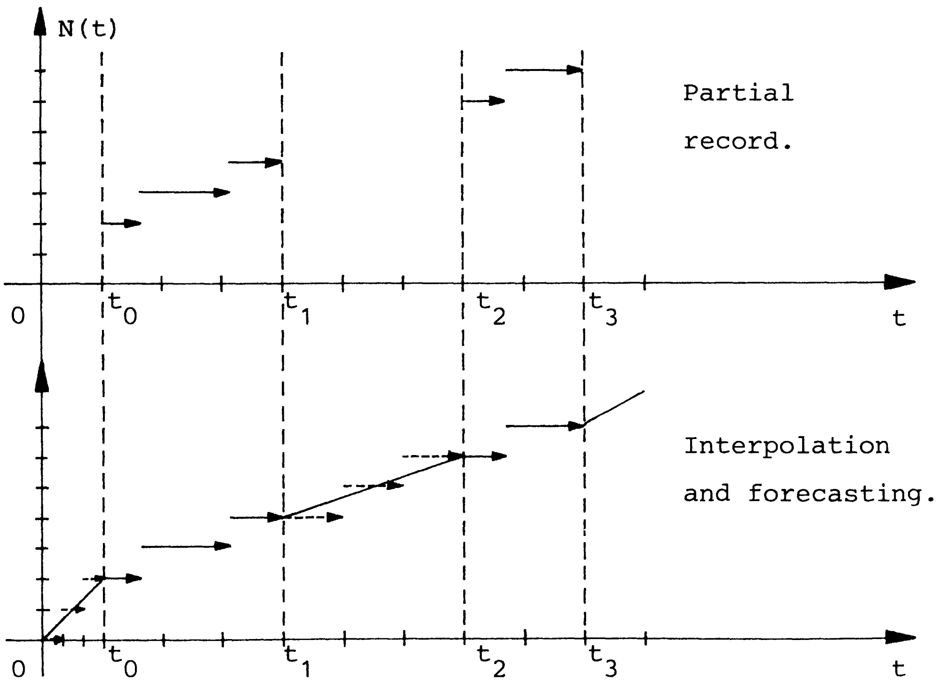
The problems of interpolation and forecasting in a Poisson's Process $N(t)$ of one parameter t, have -as the similar problems in the Brownian Functions [1]- simple solutions because of the total order in the parameter space T.

Let's recall that in the homogeneous Poisson's processes $N(t_2)-N(t_1)$ is a Poisson random variable with $\lambda=t_2-t_1$, and, in the general processes $\lambda=F(t_2)-F(t_1)$, where $F(t)$ is the distribution function of a measure on T.



For $t_1<t<t_2$, and given $N(t_1)=n_1$ and $N(t_2)=n_2$; $N(t)-N(t_1)$ is a binomial variable with $n=n_2-n_1$ and $p= =t-t_1/t_2-t_1$ (in the homogeneous process) or $p=F(t)-F(t_1)/ F(t_2)-F(t_1)$ (in the general process). So we have

$$E[N(t)-N(t_1)/N(t_1)=n_1,N(t_2)=n_2]=\frac{t-t_1}{t_2-t_1}\cdot(n_2-n_1) ,$$

or

$$E[N(t)-N(t_1)/N(t_1)=n_1,N(t_2)=n_2]=\frac{F(t)-F(t_1)}{F(t_2)-F(t_1)}\cdot(n_2-n_1).$$

This function is linear in $N(t_1)$ and $N(t_2)$.

For $t>t_3$, given $N(t_3)=n_3$; $N(t)-N(t_3)$ is a Poisson's variable with $\lambda=t-t_3$ or $\lambda=F(t)-F(t_3)$; and, then

$$E[N(t)-N(t_3)/N(t_3)=n_3]=t-t_3 \ [\text{or } F(t)-F(t_3)] \ .$$

This interpolation is not always useful because it takes continous values. A way of having integer values is to study, for the period t_2-t_1, the random time for the n_2-n_1 events, which are independent and have uniform distributions. The estimations of these instants divide t_2-t_1 into equal parts (in the homogeneous process).

POISSON'S PROCESSES OF TWO PARAMETERS.

Let's recall that a Poisson's Process $N(x,y)$ in the parameter space $E=[0,1]\times[0,1]$has the properties:

- $N(0,y)=N(x,0)=0$.
- $N(x,y)$ is a Poisson's random variable with $\lambda=x.y$.

We may consider the process defined in the parameter space of the Borel sets S of E:

- $N(S)$ is a Poisson's random variable with $\lambda=m(S)$, where $m(S)$ is a measure on E.

- If $S_1 \cap S_2=\phi$, then $N(S_1)$ and $N(S_2)$ are independent.

- If $S_1, S_2, \ldots, S_m$ is a partition of E, then

$$E[N(S)/N(S_1),N(S_2),\ldots,N(S_m)] = \sum_{i=1}^{m} E[N(S\cap S_i)/N(S_i)] =$$

$$= \sum_{i=1}^{m} N(S_i) \frac{m(S\cap S_i)}{m(S_i)}$$

because $P[N(S\cap S_i)=k/N(S_i)=n_i]$ has a binomial law with $p=m(S\cap S_i)/m(S_i)$ and $n=N(S_i)$.

ESTIMATION IN POISSON'S PROCESSES OF TWO PARAMETERS.

1.- If $N(x_i,y_i)=n_i$ are known in a finite family of points and if this family is closed for $(x_i\wedge x_j,\ y_i\wedge y_j)$; then, as a consequence of the partitions property,

$$E[N(x,y)/N(x_1,y_1),\ldots,N(x_n,y_n)] = E[N(x,y)/N(S_1),\ldots,$$

$$N(S_m)] = \sum_{i=1}^{m} N(S_i) \frac{m\{S_i\cap([0,x]\times[0,y])\}}{m(S_i)} + m([0,x]\times[0,y]-\beta)$$

where $\beta=S_1\cup S_2\cup \ldots \cup S_m$.

We can write this equations in the form

$$E[N(x,y)/N(x_1,y_1,\ldots,N(x_n,y_n)] =$$

$$= \sum_{j=1}^{n} \lambda_j N(x_j,y_j) + (x.y- \sum_{j=1}^{n} \lambda_j x_j y_j),$$

where the coeficients λ_j are calculated -as in the Brownian Functions [1]- in a linear system obtained from the Wiener-Hopf equation.

2.- Given two points (x_0,y_0), (x_1,y_1) non comparable for the partial order in R^2, the estimation of $N(x,y)$ is not linear in $N(x_0,y_0)$ and $N(x_1,y_1)$, because we do not know the values of $N(S)$ for a partition.

If, for instance, $x_0<x_1$ and $y_1<y_0$, then

$$E[N(x,y)/N(x_0,y_0)=n_0,\ N(x_1,y_1)=n_1] =$$

$$= \frac{\sum_{k=0}^{n_0\wedge n_1} k.P[B(y_0/y_1,n_0)=k].P[B(x_0/x_1,n_1)=k].\frac{k}{(x_0y_1)^k}}{\sum_{k=0}^{n_0\wedge n_1} P[B(y_0/y_1,n_0)=k].P[B(x_0/x_1,n_1)=k].\frac{k}{(x_0y_0)^k}},$$

where $B(p,n)$ is a binomial random variable.

3.- Given the values of the process in the points $(1,y)$, for each $y\epsilon[0,1]$, we know the number of events and their coordinates $y_1,y_2,\ldots,y_n$. The abcises $x_1,x_2,\ldots,x_n$ are independent and uniformly distributed random variables:

$$E[N(x,y)/N(1,\eta),\ \eta\epsilon[0,1]]=x.N(1,y).$$

This estimation is linear and coincides with the result obtained for Brownian Functions.

4.- Given $N(1,y)$ for $y\epsilon[0,1]$ and $N(x,1)$ for $x\epsilon[0,1]$, we have:

$$E[N(x,y)/N(1,\eta),\eta\epsilon[0,1];N(\xi,1),\xi\epsilon[0,1]] =$$

$$= \sum_{k=0}^{N(x,1)\wedge N(1,y)} \frac{k.\binom{N(x,1)}{k}\binom{N(1,y)}{k}}{k\,[N(x,1)-k]\,[N(1,y)-k]\,N(1,1)}.$$

This result do not coincide with the linear estimation in the Brownian Functions, given by

$$x.N(1,y)+y.N(x,1)-xy.N(1,1).$$

BIBLIOGRAPHY

[1]. E. BONET and D. NUALART. "Interpolation and Forecasting in Brownian Functions of two parameters", Stochastica, Vol. 1, 1975.

[2]. J. NEVEU. "Processus Ponctuels", Ecole d'Eté de Calcul des Probabilités de Saint-Flour, 1976.

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