Stochastica, Vol.2. 1977

THE REDFIELD TOPOLOGY ON SOME GROUPS OF CONTINUOUS FUNCTIONS.

by

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ABSTRACT

The Redfield topology on the space of real-valued continuous functions on a topological space is studied. (We call it R-topology for short). The R-neighbourhoods are described relating them to the connectedness for the carriers. The main results are:

If the space is totally disconnected without isolated points the R-topology is indiscrete. Under suitable conditions on the space R-convergence implies pointwise or uniform convergence. Under some restrictions R-convergence for a net implies that the net be eventually pointwise constant. For better behaved spaces we show that the only R-convergent sequences are the almost constant ones. In spite of corollary 5.2 of [1] we give a direct proof for the Redfield topology to be not discrete. We finally remark that for some spaces the R-topology is not first countable.

1. INTRODUCTION

We use notation and terminology of [1] , where the Redfield topology for a lattice-ordered group is studied. For the sake of completeness we recall the basic definitions.

Let G be a lattice-ordered group, G^+ the set of its non-negative elements and G_o^+ the set of its positive elements.

Let $g \varepsilon G_o^+$ and define

$I(g) = \{h \varepsilon G^+ \mid h'$ such that $h \vee h' = g,\ h \wedge h' = 0\}$

Call g indiscomposable if $I(g) = \{0,g\}$ .The set of non decomposable elements different from zero is denoted by $\mathcal{E}$

A sequence (g_n) is called a contraction sequence for $g\varepsilon$ if 1) $g_n \varepsilon$, 2) $g_1 = g$, 3) $g_{n+1} + g_{n+1} \leq g_n$ and 4) $g_n \varepsilon\, g_{n+1}$ (or equivalently 4) $g_n = g_{n+1}$).

Let $D = \{g \varepsilon \mathcal{E} \mid$ there exists a contraction seq. for g$\}$ and let

$D_2 = \{h \varepsilon G_o^+ \mid [0,h] = \{0,h\}\}$

Define for $g \varepsilon G_o^+$

$N(0,g) = [-g,g] + g$

$N_1(0,g) = \{N(0,g) \mid g \varepsilon D_1\}$

$N_2(0,g) = \{g \mid g \varepsilon D_2\}$

$N_3(0,g) = N_1(0,g) \cup N_2(0,g)$.

If $D_1 \cup D_2 \neq \phi$ put

$N(0) = \{ \bigcap_1^n H_i \mid H_i \varepsilon N_3(0,g);\ i = 1,\dots,n\}$

If $D_1 \cup D_2 = \phi$ put

$N(0) = G$.

N(0) is then a filter base and there is a unique group-topology $\mathcal{H}$ such that the filter generated by N(0) is the set of $\mathcal{H}$-neighbourhoods of zero. We call that topology the Redfield topology on G.

2.- THE REDFIELD NEIGHBOURHOODS IN SPACES OF CONTINOUS FUNCTIONS.

We first establish some elementary facts about orthogonals and non decomposable elements in the 1-group C(E) of all real valued continous functions on a topological space E.

From now on we denote by E a fixed topological space. If $g \in C(E)$, Z(g) is the zero-set of g, that is $Z(g) = g^{-1}(0)$ and C(g) is the carrier of g, i.e., $C(g) = \{x \in E \mid g(x) \neq 0\}$. We have $E = C(g) \cup Z(g)$, C(g) is open, Z(g) is closed and the union is disjoint.

PROPOSITION 2.1.-

Let $g \in C(E)$, g positive. Then g is non decomposable if C(g) is a connected set.

Proof:

If g is decomposable there exists two positive functions g_1, $g_2 \in C(E)$ such that $g_1 \quad g_2 = g$, $g_1 \quad g_2 = 0$. Therefore $C(g) = C(g_1) \cup C(g_2)$ with $C(g_i) \neq \phi$; and open.

Suppose now that C(g) is not connected. Being open one can find two disjoint non void open sets E_1, E_2 such that $C(g) = E_1 \cup E_2$. Define $g_i = g \cdot 1_{E_i}$. The functions g_i are continous and satisfy $g_1 \; g_2 = g$, $g_1 \; g_2 = 0$.

PROPOSITION 2.2

Let $g \in C(E)$, g nonnegative. If $g \neq \{0\}$ then $Z(\mathring{g}) \neq \phi$.

Proof: Take $f \in g$, $f \neq 0$. There exists $x_o \in E$ such that $x_o \in C(f)$ and for some neighbourhood V of x_o we have $V \subset C(f)$.

From proposition 2.2 we deduce inmediately

COROLLARY 2.3

Let E be completely regular. If $Z(\mathring{g}) \neq \phi$ and g is non-negative, then $g \neq \{0\}$.

COROLLARY 2.4

Let E be completely regular. The two following conditions are equivalent

1) $\mathring{Z(g)} \neq \phi$ 2) $g \neq \{0\}$

The complete regularity is not essential for the former equivalence to hold. The space constructed by E. Hewitt in [5] is completely regular but the equivalence 2.4. holds.

From the later propositions and the fact that C(E) is divisible we obtain

$D_1 = \{g \in C(E) \mid g$ is positiv and $C(g)$ connected $\}$

$D_2 = \phi$

PROPOSITION 2.5

Let $g\varepsilon D_1$. Then $f\varepsilon N(0,g)$ if $f|f(x)| \leq g(x)$ for every $x\varepsilon C(g)$.

Proof:

If $f\varepsilon N(0,g)$ we have $f = f_1+f_2$, $f_1\varepsilon[-g,g]$, $f_2\varepsilon g$, so we obtain $f = f_1$ on $C(g)$ and $-g(x)\leq f(x)\leq g(x)$. Conversely, let f be such that $|f(x)| \leq g(x)$ for $x\varepsilon$ $C(g)$. On the boundary $\partial C(g)$ is g=0 and by continuity one obtains $|f(x)|\leq g(x)$ on $\partial C(g)$ and we can define f_1 =f on $C(g)\cup \partial C(g)$, f_1= 0 on $\mathring{Z}(g)$. f_1 is continous and $f-f_1\varepsilon g$.

COROLLARY 2.6

If C(g) is dense and connected in E, then $N(0,g) = [-g,g]$.

3.- THE REDFIELD TOPOLOGY ON SOME SAPCES OF CONTINUOUS FUNCTIONS.

PROPOSITION 3.1

Let E be a totally disconnected space with no isolated points. The Redfield topology on C(E) is indiscrete.

Proof:

If $g\varepsilon D$ it must have a connected carrier, hence $C(g) =\{x\}$. But x is adeherent to $E-C(g)=Z(g)$ because x is not isolated.Hence $g(x)=0$ and $D_1 = \phi$.

For example C(Q) has indiscrete Redfield topology as well as C(K), where K is the Cantor set in $[0,1]$.

We remark that the condition on E of having no isolated points is essential, for if E has discrete topology C(E) is R^E

and corollary 4.4 of [1] applies. Then the Redfield topology on C(E) is the product topology.

LEMA 3.2

If E is such that its components E_j are open then given E_i and $\varepsilon > 0$ there exists $g_i \varepsilon$ C(E) such that $g_i | E_i =$ and $Z(g_i) = E-E_i$.

Proof: Obvious.

POROPOSITION 3.3

Let E be a topological space whose components are open. If a net f_α converges to zero (Redfield) $f_\alpha \overset{R}{\to} 0$, then $f_\alpha \to 0$ pointwise.

Proof:

Given $x \varepsilon E$ and $\varepsilon > 0$ we have $x \varepsilon E_i$ for some i and by 3.2 we can get $g_i \varepsilon$ C(E) verifying $g_i | E_i = \varepsilon$ $Z(g_i) = E-E_i$. But $g_i \varepsilon D_1$ and so $N(0,g_i)$ is a neighbourhood of zero for the Redfield topology. There exists α_o such that $\alpha \geq \alpha_o$ implies $f_\alpha \varepsilon$ $N(0,g_i)$ and using 2.5 we conclude $|f\ (x)| \leq \varepsilon$ for $\alpha \geq \alpha_o$

PROPOSITION 3.4

Let E have a finite number of connected components, each of which is open. Then $f_\alpha \overset{R}{\to} o$ implies $f_\alpha \to o$ uniformly, for every net (f_α).

Proof:

Write $E = E_1 \cup E_2 \cup \ldots\ldots \cup E_n$ where the E_i are the connected components. Given $\varepsilon > 0$ find g_i such that $g_i | E_i = \varepsilon$,

$Z(g_i) = E-E_i$ and form the zero neighbourhood

$$V= \bigcap_1^n N(0,g_i)$$

Now there exists α_o such that $\alpha\geq\alpha_o$ implies $f_\alpha\varepsilon V$, hence $f_\alpha \varepsilon N(0,g_i)$ $i=1,..,n$. But if $x\varepsilon E$ the $x\varepsilon E_i$ for some i, therefore $|f_\alpha(x)|\leq \varepsilon$ for all $x\varepsilon E$ and $\alpha\geq\alpha_o$.

Obviously the hypothesis of 3.4 are verified when E is connected or when E is locally connected and compact.

We remark that the hypotesis of 3.3 cannot be weakened, as the following example shows:

Let E be the subspace of real numbers defined by

$E= \{x\varepsilon R|\ x=0 \text{ or } x= 1/n,\ n\varepsilon N\}$

The only connected sets are the singletons and any contınous function which is zero on $\{1/n|\ n\varepsilon N\}$ must be zero on all E. Hence in this case

$$D_1 = \{f\ \varepsilon C(E) \text{ there exist } n\varepsilon N \text{ and } r\varepsilon R_o^+ \text{ such that } f= r.1_S \text{ where } S =\{ 1/n\}\}.$$

It is then obvious that $f_\alpha \xrightarrow{R} 0$ iff $f_\alpha(x) \to 0$ for every $x\neq 0$.

PROPOSITION 3.5

Let E be such that its connected components are open, then the Redfield topology on C(E) is Hausdorff.

Proof:

We apply proposition 5.6 of [1] . Let $f\varepsilon C(E)$, $f>0$. Take $x\varepsilon E$ such that $f(x)=2\varepsilon> 0$. If K is the connected component where x lies we can get a function g such that $g|K =\varepsilon$ and $Z(g)=E-K$. It is then obvious that $f\notin N(0,g)$.

PROPOSITION 3.6

Let E satisfy

1) The points are zero-sets.

2) If x is not isolated, then there exist two sets E_1 and E_2, with E_1 open, connected and not empty and E_2 open such that $E-\{x\} = E_1 \cup E_2$ and $x \varepsilon \overline{E}$.

Then if $f_\alpha \overset{R}{\to} 0$ and x is not isolated, f_α (x) is eventually null.

Proof:

Let x_o be not isolated and put $E-\{x_o\} = E_1 \cup E_2$, where E_1 and E_2 satisfy the conditions 2). There exists a non-negative $g\ \varepsilon C(E)$ such that $Z(g) = \{x_o\}$. Define $g_1 = g\ 1_{E_1}$. One can easely verify that g_1 is continuous and $C(g_1) = E_1$, so that $g_1 \varepsilon D_1$ and therefore $N(0,g_1)$ is a Redfield neigbourhood of zero. There exists α_o such that for $\alpha \geq \alpha_o$ we have $f_\alpha \varepsilon\ N(0,g_1)$, that is

$$f_\alpha\ \varepsilon[-g_1,\ g_1] \ + g_1 \quad \text{if} \quad \alpha \geq \alpha_o$$

Hence $f_\alpha = g_\alpha + f^*_\alpha$, $g_\alpha \varepsilon\ [-g_1, g_1]$, $f^*_\alpha \varepsilon\ g_1$

Now f^*_α vanishes on E_1 and x_o being adherent to E_1 we must have $f^*_\alpha\ (x_o) = 0$. Hence

$$f_\alpha(x_o) = g_\alpha(x_o) + f^*_\alpha\ (x_o) = 0$$

Naturally 3.6 applies when $E=R^n$.

The following result can be proved using proposition 2.35 of Redfield´s Doctoral dissertation and applying 5.6 of [1] to show that R is Hausdorff. Nevertheless we give a direct proof as a corollary of 3.6.

COROLLARY 3.7

Suppose that E satisfies the requierements of 3.6. Then the Redfield neigbourhoods of N(0) as defined in the introduction are closed.

Proof:

It will suffice to show that each N(0,g) is closed.

Let $f_\alpha \overset{R}{\to} f$, $f_\alpha \varepsilon$ N(0,g). One must show $f\varepsilon$ N(0,g). But $f_\alpha - f \overset{R}{\to} 0$. Take $x\varepsilon C(g)$. Now $f_\alpha(x)-f(x)$ shall be eventually null, so for $\alpha \geq \alpha_o$, $f_\alpha(x)=f(x)$, that is, $|f_\alpha(x) \leq g(x)$, so $|f(x) \leq g(x)$ and the Proposition follows.

LEMA3.8

Let E be a perfectly normal space, F closed in E and f: F $\to$ R continous, $f \geq 0$. There exists $f^*\varepsilon C(E)$ such that

1) $f^* |F = f$ 2) $Z(f^*) = Z(f)$

Proof:

E is normal, and using Tietze´s theorem one can get g $\varepsilon C(E)$ such that $g|F= f$ and g is also nonnegative. Being E perfectly normal F is a zero-set and there exists a function h $\varepsilon C(E)$ with $Z(h)=F$ (we can take h nonnegative).Define $f^* = g+h$ and one easily sees that f^* meets the requirements.

PROPOSITION 3.9

Let E be

1) connected

2) perfectly normal

3) first countable

4) such that if $x\in E$, x^c can be written as a finite disjoint union of open connected sets.

5) countable compact.

Then if the sequence $f_n \overset{R}{\to} 0$, f_n is eventually null.

Proof:

We can take f_n nonnegative. Switching down to a subsequence we can suppose $\|f_n\|_\infty \leq M$, for E cannot have isolated points (otherwise it would not be connected) and we can apply 3.4. Suppose that (f_n) is not eventually null. We can find points (x_n) such that $f_n(x_n)= \varepsilon_n \neq 0$. Now (x_n) cannot have an almost constant subsequence because conditions 1) and 2) of 3.6 are verified and the existence of such a subsequence would contradict 3.6., hence changing to a subsequence we can suppose all (x_n) different. Again, as (ε_n) is bounded, and considering another subsequence we may suppose that (ε_n) is convergent. Finally, using sequential compactness, we can select a convergent subsequence (y_n) of (x_n). After all choices we get subsequence (g_n) of (f_n), (y_n) of (x_n), (r_n) of (ε_n), such that

i) $y_n \to y_o$. ii) $r_n \to r_o$. iii) All y_n are different.

iv) $g_n(y_n)=r_n$

There are two possible cases

1) $r_o \neq 0$. We then consider the closed set $F= \{y_n, y_o\}$ (E is first countable) and the function $f:F \to R$ defined by

$f(y_n)=2^{-1}r_n$, $f(y_o)=2^{-1}r_o$, f is well defined because of iii).

It is also continuous an F, and by Lemma 3.8 there exists a function f^* such that $f^* \mid F=f$ and $Z(f) = Z(f)=\phi$ .The carrier of f^* is all E and being connected, $f^* \varepsilon D_1$. The functions g_n do not belong to the neighbourhood $N(0,f)$, for $g_n(y_n)=r_n$ and so $g_n \notin [-f,f]$ (we are using 2.5).

2) $r_o=0$. Then $E-\{y_o\} = E_1 \cup E_2 \cup \ldots \cup E_n$, every E_i open and connected. We again consider the closed set $F=\{y_n,y_o\}$ and the function $f(y_n) = 2^{-1}r_n$, $f(y_o)=0$ continuous and well defined on F. We again use Lemma 3.8 and find a continuous extension f^* to all E, verifying $Z(f^*)=\{y_o\}$ i.e.

$$C(f^*)= E-\{y_o\}$$

The functions $f_i= f\, 1_{E_i}$ can be shown to be continuous by an argument similar to that of 2.1, hence $C(f_i)=E_i$ and $f_i \varepsilon D_1$, so that the set

$$V= \bigcap_1^n N(0,f_1)$$

is a Redfield neighbourhood of zero in $C(E)$. Now no g_n belongs to V, for given $n \varepsilon N$, we have $y_n \varepsilon E_i$ for some i. If $g_n \varepsilon N(0,f_i)$, then $g_n(y_n) \leq 2^{-1} r_n$, a contradiction.

Proposition 3.9 can be applied, obviously, when E is a compact,non void interval in R^n. Nevertheless the countable compactness (equivalent to sequential compactness in presence of first countability) can be avoided. When E is all R it can be shown, using local compactness, that the result still holds. In general when E verifies conditions 1) and 2) of 3.9 and is also metrizable, locally and σ-compact one can show that the only Redfield-convergent sequences are the almost constant ones.

We see, in view of 3.9 that if E verifies the requeriments thereof, then the Redfield topology in C(E) is not first countable because corollary 5.2. of [1] assures that the Redfield topology in C(E) is not discrete, and having no convergent sequences other than the almost constant it cannot be first countable.

For the sake of completeness we give an additional proof for the non-discreteness of the Redfield topology on C(E).

PROPOSITION 3.10

Let E be a perfectly normal and locally compact space. Then the Redfield topology on C(E) is not discrete.

Proof:

Let V be a Redfield neighbourhood of zero. Then

$$V= \bigcap_{1}^{n} N(0,g_i)$$

Put $I_i = C(g_i$, $i=1,..,n$. Reordering I_i if necessary we can find $k \leq n$ such that $A= \bigcap_{1}^{k} I_i \neq \phi$, and $A \cap I_j = \phi$ if $j=k+1,..,n$.

Now A is open and there exists a compact neigbouhood K such that $x \varepsilon K \subset A$, for some x E. Let $m_i = \min_K g_i$ and $m=\min_i m_i \neq 0$. Let U be an open set containing x and contained in K; E-U is closed and therefore is a zero-set. Take f vanishing exactly on E-U and nonnegative; f is then bounded on K and hence an all E. Multiplying if necessary by a constant we may suppose $f \leq m$, that is $f \leq g_i$, $i=1,..,k$. But C(f) is disjoint from $I_{k+1},..,I_n$ and obviously $f \varepsilon N(0,g_i)$, $i=k+1,..,n$ and also $f \varepsilon N(0,g_i)$ $i= 1,..,k$.

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