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ON NATURAL METRICS

by

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0. INTRODUCTION.

Karl Menger defined in (1) the concept of natural metric of an abelian group (G,+), assigning to every pair of elements p,q in G the unordered pair (p-q,q-p).

D.O. Ellis in (2) studied the geometric caracter of "free-mobility" of that structure, and one of us (3) has given an interpretation of this concept in the theory of generalized metric spaces (4). In the present note we study the effective construction of a natural generalized metric structure (on a set), obtaining as particular case the result of Menger. In the case of groups we analyze its topology and its structure of natural proximity space (in the sense of Efremovič).

1. ON THE NATURAL METRIC OF A SET

A generalized metric space is a tripled (Ω,S,d) where Ω is a non empty set, S is an ordered semigroup with neutral e and, occasionaly, minimum o, and $d:\Omega \times \Omega \to S$ verifies:

a) $d(X,Y) = e \Leftrightarrow X=Y$.

b) $d(X,Y) = d(Y,X), \forall X,Y \in\Omega$.

c) $d(X,Y) \leq d(X.Z)+ d(Z,Y), \forall X,Y,Z \in \Omega$

Given (Ω,S,d) and (Ω',S',d'), a morphism between them (vid.(4)) is a pair (f,g) where $f:\Omega \to \Omega'$, $g:S \to S'$ is a morphism-semigroup and such that $d' \circ f \times f = g \circ d$. A morphism (f,g) is an isomorphism if f is bijective and g is an isomorphism of semigroups.

The more suggestive examples are the ordinary metric spaces of Frèchet (Ω,R,d), the P.M.S., of Wald (Ω,Δ^+,F) where Δ^+ is the set of positive distribution functions, with convolution, and the boolean space $(B,B,\triangle)$ of a Boole's algebra with symmetric difference $\triangle$ (vid(5)). Let $(G,+,e)$ be an abelian group with neutral e. If $P_F(G)$ is the set of finite subsets of G, we define the point-set operation:

$$A \oplus B = \{a + b \ ; \ a \in A,\ b \in B\},\ \forall A,B \in P_F(G),$$

therefore $(P_F(G),\oplus,\subset)$ is a commutative ordered semigroup with neutral $\{e\}$, and minimum ϕ. Given a set $\Omega \neq \phi$ and a fixed function $f:\Omega \to G$, consider the mapping

$$d_f:\Omega \times \Omega \to P_F(G)$$
$$(X,Y) \to \{ f(X)-f(Y),\ f(Y)-f(X) \}$$

which verifies the three properties (a) $d_f(X,X) = \{e\}$;

(b) $d_f(X,Y) = d_f(Y,X)$; (c) $d_f(X,Y) \subset d_f(X,Z) \oplus d_f(Z,Y)$.

We obtain in this way a structure of generalized metric space for Ω, $(\Omega,P_F(G),d_f)$, which is trivial when f is constant and separated when f is one-to one (note that if Ω is a semigroup with zero divisors, for example $Z/_{(2n)}$, there is no one-to-one application into a group and then Ω is not a separated space).

We consider the following cases:

a) If $\Omega = G$ and $f=I_G$, then $d_{I_G}(X,Y) = \{X-Y, Y-X\}$ is the natural metric $(G,P_F(G),d_{I_G})$ associated naturally to every abelian group (vid(1)). In particular if G is an involutive group, $d_{I_G}(X,Y)= \{X + Y\}$ (vid (3)).

b) If $\Omega = (G',.)$ is a non-abelian group, taking $(G,+) = = (\frac{G'}{D(G')}, F)$

(the abelian group obtained from G' by passing to the quotient of the derivate $D(G')= \{X.Y.X^{-1}.Y^{-1};\ X,Y\varepsilon\ G'\}$), and taking $f:G'\to G'/_{D(G')}$, $f(x)=\overline{X}$, the natural projection, we extend the concept of natural metric to the non **abelian** groups: $(G', P_F(G'/_{D(G')}), d_f)$.

c) If $\Omega = (S,+)$ is an abelian semigroup with neutral which is a quasi-group, there is a one-to-one mapping into a group $\overline{S}$, $f:S \to \overline{S}$, $f(X) = \overline{X}$, inducing the metric structure $(S,P_F(\overline{S}),d_f)$.

d) If $\Omega = (G,\vee,\wedge,+)$ is a Riesz´s group (L-group), taking $f=I_G$, we obtain $(G,P_F(G),d_{I_G})$ and there exist a "similitude" morphism $(I_G;\ Max)$ wich $(G,G^+,|\ |)$ (vid(4)), the next diagram is commutative:

$$\begin{array}{ccc} G \times G & \xrightarrow{d_{I_G}} & P_F(G) \\ & \searrow^{|\ |} & \downarrow \text{Max} \\ & & G^+ \end{array}$$

therefore $(\text{Max} \circ d_{I_G})(X,Y) = \text{Max}\{X-Y, Y-X\} = (X-Y) \vee (Y-X) =$
$= |X-Y|$.

That is the case of the usual numerical sets, when $G = N, Z, Q, R$ or C and $|\ |$ is the absolute value:

$$\begin{array}{ccccc} G \times G & \xrightarrow{d_{I_G}} & P_F(G) & \xrightarrow{\text{Max}} & R \\ (X,Y) & \longmapsto & \{X-Y, Y-X\} & \longmapsto & |Y-X|. \end{array}$$

e) If $\Omega \neq \phi$ and $G' = (P(\Omega), \cup, \cap, \Delta)$ results the trivial metric $d(X,Y) = \{\{X\} \ \Delta \ \{Y\}\}$.

PROPOSITION 1.

If two groups $(G,+)$, $(G',.)$ are isomorphic, the metric structures on $(\Omega, P_F(G))$ and $(\Omega, P_F(G'))$ are isomorphes.

Proof. Let $H: G \to G'$ be the isomorphism. Given $f \in G^{\Omega}$, let $g = H \circ f \in G'^{\Omega}$. The spaces $(\Omega, P_F(G), d_f)$ and $(\Omega, P_F(G'), d_g)$ **are metrically** isomorphes because $d_g(X,Y) =$
$= \{g(X).g(Y)^{-1}, g(Y).g(X)^{-1}\} = \{H(f(X))-H(f(Y)), H(f(Y))-$
$- H(f(X))\} = \hat{H}(\{f(X)-f(Y), f(Y)-f(X)\}) =$
$= (\hat{H} \circ d_f)(X,Y)$, $(X,Y) \in \Omega \times \Omega$, and so $(I_\Omega, \hat{H})$ is a

morphism with $\hat{H}$ isomorphism between $P_F(G)$ and $P_F(G')$.□

PROPOSITION 2.

$f:(G,+,e) \to (G',.,u)$ is a group-morphism, if and only if, $\hat{f} \circ d_{I_G} = d_{I_G} \circ fXf$, that is, if the next diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{d_{I_G}} & P_F(G) \\ {\scriptstyle f \times f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ G' \times G' & \xrightarrow{d_{I_{G'}}} & P_F(G') \end{array}$$

Proof. If f is a morphism, for every pair $X,Y \in G$ is

$(d_{I_{G'}} \circ f \times f)(X,Y) = \{f(X).f(Y)^{-1}, f(Y).f(X)^{-1}\}=$
$= \{f(X-Y), f(Y-x)\} = \hat{f}(\{X-Y, Y-X\}) = (\hat{f} \circ d_{I_G})(X,Y)$. Reciprocally, if $\hat{f} \circ d_{I_G} = d_{I_{G'}} \circ fXf$, there results $\{f(X-Y),f(Y-X)\}=$ $= \{f(X).f(Y)^{-1}, f(Y).f(X)^{-1}\}$ and therefore (a) $f(X-Y) = f(X).f(Y)^{-1}$ and $f(Y-X) = f(Y).f(X)^{-1}$ or (b) $f(X-Y)=f(Y).f(X)^{-1}$ and $f(Y-X) = f(X).f(Y)^{-1}$. In the case (a), taking X=Y we have $f(e)=u$ and $X = e$, $f(-Y)=f(e).f(Y)^{-1} = f(Y)^{-1}$ and then $f(x-y)=f(X).f(-Y)$ that is, f is morphism. In the case (b) we proceed identically.□

COROLLARY 1.

Two groups are isomorphic, if and only if, they are metricaly isomorphic with respect the associated natural metric structures. □

Given the metric structure $(S,P_F(G), d_\pi)$ of a quasigroup S injectable by π into the group $(G,+,e)$, if $e\in H\subset S$, let

$M(H) = \{\alpha:H\to G/d_{I_G} \circ \alpha x\alpha = d_\pi, \alpha$ one-to-one be the set of "congruences" between H and some subset of G. If $\alpha\in M(H)$, then for every $x\in H$ we have $\{\pi(x)-\pi(o),\pi(o)-\pi(x)\} =\{\alpha(x)-\alpha(o),\alpha(o)-\alpha(x)\}$,

that is, $\alpha(x)=\pi(x)+\alpha(o)$ or $\alpha(x)=-\pi(x)+\alpha(o)$. If $H=S=G$, $M(G)$ is the natural metric geometry of G and its elements are called the isometries of $(G,P_F(G),d_{I_G})$.

The concept of metric "betweennes" has not sense in the natural metric structure because the Menger´s axioms are not satisfied.

The category of groups having a product, we can extend the idea of natural metrization over the product of a family of sets $(\Omega_i)_{i\in I}$ (where I is a directed set) with a family

$(G_i)_{i\in I}$ of groups and $f_i: \Omega_i\to G_i$, $i\in I$. We define

$d_{\pi_{i\in I} f_i} : (\pi_{i\in I}\ \Omega_i)^2 \to P_F(\pi_{i\in I}\ G_i)$, $d_{\pi_{i\in I} f_i}((X_i)_{i\in I},(Y_i)_{i\in I})=$

$= \{(f_i(X_i)-f_i(Y_i))_{i\in I}, (f_i(Y_i)-f_i(X_i))_{i\in I}\}$ and we obtain

the natural metric product $(\pi_{i\in I}\ \Omega_i, P_F(\pi_{i\in I}\ G_i), d_{\pi_{i\in I} f_i})$.

2. PROXIMITY STRUCTURE ON $(G, P_F(G), d_{I_G})$.

In (4) the standard topologies associated with a generalized metric space (Ω,S,d) are T_d and T_d^*.

If the metric structure is $(G, P_F(G), d_{I_G})$, then for every $P\varepsilon G$ and $R\varepsilon P_F(G)$ with $e\varepsilon R$, the neighborhood of center p and radius R is $E_p(R) = \{q\varepsilon G | \{p-q, q-p\} \subset R\}$.

The $T^*_{d_{I_G}}$ is the minimum topology over G that contains the family $E = \{E_p(R);\ p\varepsilon G,\ R\varepsilon P_F(G),\ e\varepsilon R\}$ and $T_{d_{I_G}}$ is the family formed by ϕ and those $A\varepsilon P_F(G)$ such that "for every $p\ \varepsilon A$, exist $E_p(R) \subset A$" . In our case $T_{d_{I_G}} = T^*_{d_{I_G}} = P(G)$ for, $X\ \varepsilon G$, $\{X\} = E_X(\{e\})$.

Having a non-trivial topology in $(G,P_F(G),d_{I_G})$, we define on P(G) a binary relation as follows:

$A,B\ \varepsilon P(G):\ A\ \delta B \Leftrightarrow (A\cap B\neq\phi \text{ or } (A\cap B=\phi \quad a\varepsilon A,\ \ b\varepsilon B |\ a=-b))$.

It is easily verified:

(1) $A\ \delta B \Rightarrow A\neq\phi$, $B\neq\phi$; (2) $A\cap B \neq\phi\Rightarrow A\delta B$; (3) $A\delta B\Rightarrow B\delta A$;

(4) $A\ \delta(B\cup C)\Leftrightarrow A\delta B$ ó $A\delta C$.

And furthermore the "strong-axiom" (5) $A\not\delta B\Rightarrow E\varepsilon P(G)$ such that $A \not\delta E$ and $(G-E)\not\delta B$. Obviously $A\not\delta B$ implies $A\cap B=\phi$ and $a\neq-b$, $(a,b)\varepsilon AxB$; defining $E= B\cup\{-b;\ b\varepsilon B\} \supset B$ we obtain $A\not\delta E$ and $(G-E)\not\delta B$.

We conclude that (G,δ) is a proximity space, in the sense of Efremovic, and its associated topology has the clousure-operator $A^{\delta} = \{X \in G \mid \{X\} \delta A\} = A \cup (-A)$. Then $\{X\}^{\delta} = \{X,-X\}$ and (G,δ) is T_1, if and only if, G is involutive.

Obviously all group morphisms are δ-continous and two subgroups are δ-related and closed.

The natural metric on G studied in § 1, is

$$d_{I_G} : G \times G \qquad P_F(G)$$

where $d_{I_G}(X,Y) = \{X-Y,Y-X\} = \{X-Y\}^{\delta}$ **which maps** every pair of points in G into the smallest closed set in the δ-topology that contains its difference. Because of all that we call it the natural proximity in the group G.

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