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LOCATION OF THE 2-CENTERS OF THREE POINTS[1]

(Optimal location, medians, centers of finite sets, characterization of inner product spaces)

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ABSTRACT

We prove that a real normed space X of dimension greater or equal than 3 is an inner product space if and only if, for every three points $u, v, w \in X$, the set of points at which the function $x \in X \to \|u - x\|^2 + \|v - x\|^2 + \|w - x\|^2$ attains its minimum intersects the convex hull of these three points.

1. INTRODUCTION

Let S and S^* be the unit spheres of a real normed space X and its topological dual X^*, respectively. For $u \in X$ and $f \in X^*$, denote $Ju = \{f \in S^* : f(u) = \|u\|\}$ and $J^*f = \{u \in S : f(u) = \|f\|\}$.

A point $z \in X$ is said to be a 2-center of the points $u, v, w \in X$ when

$$\|u - z\|^2 + \|v - z\|^2 + \|w - z\|^2 = \inf_{x \in X} (\|u - x\|^2 + \|v - x\|^2 + \|w - x\|^2)$$

and it is said to be a Chebyshev center, or ∞-center, when

$$\sup(\|u - z\|, \|v - z\|, \|w - z\|) = \inf_{x \in X} \sup(\|u - x\|, \|v - x\|, \|w - x\|).$$

The set of 2-centers, the set of ∞-centers, and the convex hull of the points $u, v, w \in X$ will be denoted by $Z^2(u, v, w)$, $Z^\infty(u, v, w)$, and $\mathrm{co}(u, v, w)$, respectively.

A well-known theorem of Garkavi [4] and Klee [7] says that a real normed space X of dimension ≥ 3 is an inner product space (i.e., its norm is induced by an inner product) if and only if, for every $u, v, w \in X$,

$$Z^\infty(u, v, w) \cap \mathrm{co}(u, v, w) \neq \emptyset.$$

We shall prove that the same is true for $Z^2(u, v, w)$ instead of $Z^\infty(u, v, w)$.

2. RESULTS

Lemma 1 ([3, 5]). *Let X be a real normed space and let $u, v, w \in X$. Then $0 \in Z^2(u, v, w)$ if and only if there exist $f \in Ju$, $g \in Jv$ and $h \in Jw$ such that*

$$\|u\| f + \|v\| g + \|w\| h = 0.$$

Lemma 2. *Let X be a real normed space and let $u, v, w \in X \setminus \{0\}$ be such that $0 \in Z^2(u, v, w)$. If f, g, h are as in Lemma 1 and if $z \in Z^2(u, v, w)$, then $z \in \ker f \cap \ker g \cap \ker h$, and, for every $r, s, t \in [0, 1]$,*

$$0 \in Z^2(u - rz, v - sz, w - tz).$$

Proof. Since the function

$$r \in \mathbb{R} \to \|u - rz\|^2 + \|v - rz\|^2 + \|w - rz\|^2$$

is convex and attains its minimum at $r = 0$ and $r = 1$, it is obvious that, for every $r \in [0, 1]$, $0 \in Z^2(u - rz, v - rz, w - rz)$.

Let f, g, h be as in Lemma 1 and let $r \in [0, 1]$. Then

$$\begin{aligned}\|u - rz\|^2 + \|v - rz\|^2 + \|w - rz\|^2 &= \|u\|^2 + \|v\|^2 + \|w\|^2 = \\ &= \|u\| f(u) + \|v\| g(v) + \|w\| h(w) = \\ &= \|u\| f(u - rz) + \|v\| g(v - rz) + \|w\| h(w - rz) \leq \\ &\leq \|u\|\,\|u - rz\| + \|v\|\,\|v - rz\| + \|w\|\,\|w - rz\| \leq \\ &\leq \frac{1}{2}(\|u - rz\|^2 + \|u\|^2 + \|v - rz\|^2 + \\ &\quad + \|v\|^2 + \|w - rz\|^2 + \|w\|^2)\end{aligned}$$

and, thus, $f(u - rz) = \|u - rz\| = \|u\| = f(u)$. Therefore, $f(z) = 0$, and $f \in J(u - rz)$. The same argument can be

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used to obtain that $g(z) = 0$ and $g \in J(v - sz)$, $h(z) = 0$ and $h \in J(w - tz)$.

Hence,

$$\|u - rz\|f + \|v - sz\|g + \|w - tz\|h = 0,$$

as we wished to show (see Lemma 1). □

Lemma 3 ([2, 1, 6]). *A real normed space X of dimension ≥3 is an inner product space if and only if there is a norm-*1 *linear projection of X onto every* 2*-dimensional subspace of it.*

Proposition 1. *A real normed space X of dimension ≥3 is an inner product space if and only if, for every* $u, v, w \in X$,

$$Z^2(u, v, w) \cap \mathrm{co}(u, v, w) \neq \emptyset.$$

Proof. Although the necessary part is essentially in [8], it is easy to repeat it here. Suppose that X is an inner product space and that $x \notin \mathrm{co}(u, v, w)$. Since $\mathrm{co}(u, v, w)$ is a convex and compact set, there exists a closed hyperplane H that strictly separates x and $\mathrm{co}(u, v, w)$.

Let $y = P_H(x)$ be the orthogonal projection of x into H and let u', v', w' be the intersection with H of the straight lines that connect x with u, v, w, respectively.

Then, it follows from $u - x = u - y + y - x$, that

$$\|u - x\|^2 = \|u - y\|^2 + \|y - x\|^2 + 2(u - y | y - x)$$

where $(u - y | y - x)$ is the inner product of $u - y$ and $y - x$.

Since $u - y = u - u' + u' - y$ and $u' - y \perp y - x$, $(u - y|y - x) = (u - u'|y - x)$, but $u - u' = \rho(u' - x)$, with $\rho > 0$, and, hence, $(u - u'|y - x) = \rho(u' - x|y - x)$. Finally, it follows from $u' - x = u' - y + y - x$ and $u' - y \perp y - x$, that

$$\|u - x\|^2 = \|u - y\|^2 + \|y - x\|^2 + 2\rho(y - x|y - x) > \|u - y\|^2.$$

Analogous calculations for $v - x$ and $w - x$ yield to

$$\|u - y\|^2 + \|v - y\|^2 + \|w - y\|^2 < \|u - x\|^2 + \|v - x\|^2 + \|w - x\|^2,$$

i.e., $x \notin Z^2(u, v, w)$.

So, the continuous function $x \in X \rightarrow \|u - x\|^2 + \|v - x\|^2 + \|w - x\|^2$ attains its minimum in the compact set $\mathrm{co}(u, v, w)$, i.e., not only $Z^2(u, v, w) \cap \mathrm{co}(u, v, w) \neq \emptyset$, but $Z^2(u, v, w) \subset \mathrm{co}(u, u, w)$.

Taking into account the nature of the hypothesis and the fact that X is an inner product space if and only if so is every 3-dimensional subspace of it, to prove the converse we may suppose that $\dim X = 3$. In this way we can use the kind property that $J^* f \neq \emptyset$ for any $f \in X^*$.

Let $P = \ker k$, with $k \in S^*$, be an homogeneus plane of X and let $u, v \in S \cap P$ be two non-proportional vectors such that there are $f \in Ju$ and $g \in Jv$ whose restrictions, $f|_P$ and $g|_P$, to P are non-proportional. Then, the straight line $L = \ker f \cap \ker g$ is non-contained in P and $\{f|_P, g|_P\}$ is a basis of P^*.

We shall prove that S is supported at any $w \in S \cap P$ by the straight line $w + L$, i.e., that

$$x \in X \rightarrow (x + L) \cap P$$

is a norm-1 linear projection of X onto P (see Lemma 3).

This is equivalent to see that, for every $w \in S \cap P$, there exist $\lambda, \mu \in \mathbb{R}$ such that $\|\lambda f + \mu g\| = 1$ and $w \in J^*(\lambda f + \mu g)$, i.e.,

$$S \cap P = \bigcup_{\|\lambda f + \mu g\| = 1} J^*(\lambda f + \mu g) \cap P.$$

Since $\{f|_P, g|_P\}$ is a basis of P^*, for every $w \in S \cap P$ there are $\lambda, \mu \in \mathbb{R}$ such that

$$1 = \|\lambda f|_P + \mu g|_P\| = \sup_{y \in S \cap P} |\lambda f(y) + \mu g(y)| = \lambda f(w) + \mu g(w),$$

i.e.,

$$S \cap P = \bigcup_{\|\lambda f|_P + \mu g|_P\| = 1} J^*(\lambda f|_P + \mu g|_P).$$

Hence, we only need to show that any $\lambda f + \mu g$ such that $\|\lambda f + \mu g\| = 1$, attains its norm at $S \cap P$, i.e.,

$$\sup_{x \in S} |\lambda f(x) + \mu g(x)| = \sup_{y \in S \cap P} |\lambda f(y) + \mu g(y)|$$

or, equivalently,

$$J^*(\lambda f + \mu g) \cap P \neq \emptyset.$$

This is obvious for either $\lambda = 0$ or $\mu = 0$. Assume, by contradiction, that there are $\lambda, \mu \in \mathbb{R} \setminus \{0\}$ such that $J^*(\lambda f + \mu g) \cap P = \emptyset$ and denote $h = -(\lambda f + \mu g)$. Since J^*h is convex, changing if necessary the sign of k we can assume that $k(y) > 0$ for every $y \in J^*h$. Then, since X is finite dimensional, J^*h *is* a non-void compact subset of S and there exists $w \in J^*h$ such that

$$0 < k(w) = \inf_{y \in J^*h} k(y).$$

It follows from Lemma 1 and

$$\lambda f + \mu g + h = |\lambda|(\text{sign } \lambda)f + |\mu|(\text{sign } \mu)g + h = 0$$

that $0 \in Z^2(\lambda u, \mu v, w)$.

Since $0 \notin \text{co}(\lambda u, \mu v, w)$, there is $z \in Z^2(\lambda u, \mu v, w) \cap \text{co}(\lambda u, \mu v, w)$ and, by Lemma 2, $z \in L = \ker f \cap \ker g$, $w - z \in J^*h$. Furthermore, it follows from $z \in \text{co}(\lambda u, \mu v, w) \backslash P$, $k(u) = k(v) = 0$ and $k(w) > 0$ that $k(z) > 0$. Hence, $k(w - z) < k(w)$ which contradicts the definition of w. □

Remark 1. It is well known [8] and easy to prove that the hypothesis $Z^2(u, v, \text{w}) \cap \text{co}(u, v, w) \neq \emptyset$ is equivalent to the apparently weaker condition $Z^2(u, v, w) \cap \text{aff}(u, v, w) \neq \emptyset$, where $\text{aff}(u, v, w)$ is the affine hull of u, v, w.

Remark 2. We have seen in the well known first part of the proof of Proposition 1 that if X is a real inner product space, then $\emptyset \neq Z^2(u, v, w) \subset \text{co}(u, v, w)$. Furthermore, it is also easy to check that, in this case, $Z^2(u, v, w)$ has a unique element.

Example 1. Some straightforward calculations show that if $X = l_1^3$ and $u = (1, 0, 0)$, $v = (0, 1, 0)$, $w = (0, 0, 1)$, then $Z^2(u, v, w) = \{(0, 0, 0)\}$ and, hence, $Z^2(u, v, w) \cap \text{co}(u, v, w) = \emptyset$.

Example 2. If $X = l_\infty^2$ and $u = (1, 1)$, $v = (1, 0)$, $w = (-2, 0)$, then $Z^2(u, v, w) = \{(0, t) : 0 \leq t \leq 1\}$ and, hence, $Z^2(u, v, w) \cap \text{co}(u, v, w) \neq \emptyset$, but $Z^2(u, v, w) \not\subset \text{co}(u, v, w)$.

Remark 3. It is easy to see that if X is reflexive, then $Z^2(u, v, w) \neq \emptyset$ for every $u, v, w \in X$. However, L. Veselý has proved in [9] (not only for 2-centers, but with considerably greater generality) that if X is non-reflexive, then there are $u, v, w \in X$ and an equivalent norm in X, $|||\cdot|||$, such that $Z^2_{|||\cdot|||}(u, v, w) = \emptyset$.

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