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NONLINEAR METRIC PROJECTIONS IN TWISTED TWILIGHT

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ABSTRACT

By definition, quasi-linear maps $F\colon X \to \mathbb{R}$ on a quasi-Banach K-space X can be approximated by linear maps. We investigate the nature of the method $F \to m(F)$ that yields the almost optimal approximation, and which we call metric projection. We shall show that the method of concentrators of Kalton and Roberts that works for $\mathcal{L}_\infty$-spaces is quasi-linear and non-linear. We shall also show that there exists a linear metric projection if and only if the Banach envelope of X is an $\mathcal{L}_1$-space.

1. INTRODUCTION

Some quasi-Banach spaces X have the following nice property, for which they are awarded with the term *K-spaces:* whenever E is a quasi-Banach containing a one-dimensional subspace $\mathbb{R}$ such that $E/\mathbb{R} = X$ and $t\colon M \to X$ is an operator from another quasi-Banach space M then t can be lifted to an operator $T\colon M \to E$. Equivalently, every exact sequence $0 \to \mathbb{R} \to E \to X \to 0$ splits. Equivalently, still, every quasi-linear map $F\colon X \to \mathbb{R}$ is at finite distance from some linear map $L\colon X \to \mathbb{R}$. The preliminaries section contains precise definitions for these terms. When X is a Banach space then the preceding statements are equivalent to: whenever E is a quasi-Banach space such that $E/\mathbb{R} = X$ then E is itself a Banach space (and thus isomorphic to $\mathbb{R} \oplus X$).

The main examples of K-spaces are: B-convex Banach spaces and quasi-Banach L_p spaces, $0 < p < 1$, proved by Kalton in [17]; and the $\mathcal{L}_\infty$-spaces (proved by Kalton and Roberts in [24]). On the opposite side, the main examples of non-K-spaces are the $\mathcal{L}_1$-spaces. The three proofs have different nature. The proof for B-convex Banach spaces consists in directly showing that the so-called *twisted sum* space E is locally convex. The proof for L_p is a cunning computation that shows that given a quasi-linear map F on a finite dimensional $l_p(n)$ space the «obvious» linear map $l(e_j) = F(e_j)$ is at finite distance (independently of n) from F. However, the proof for $\mathcal{L}_\infty$-spaces is highly nontrivial.

Kalton and Roberts proved in [24] is that *If* $F : l_\infty(\Omega) \to \mathbb{R}$ *is a quasilinear map then there is a linear functional* $L : l_\infty(\Omega) \to \mathbb{R}$ *with* $|F(x) - L(x)| \leq 100\, Q(F)\, \|x\|$, *where* $Q(F)$ *is the quasilinear constant of* F. Let us give a different statement. Recall that a function $f\colon \mathcal{A} \to \mathbb{R}$ defined on an algebra of subsets of a set Ω is said to be ε-approximately additive if $f(\emptyset) = 0$ and for every pair A, B of disjoint sets one has

$$|f(A \cup B) - f(A) - f(B)| \leq \varepsilon.$$

Given a quasi-linear map $F\colon l_\infty(\Omega) \to \mathbb{R}$, then $f(A) = F(1_A)$ defines a $Q(F)$-approximately additive function on 2^Ω. Additive set-functions are the 0-approximately additive, and correspond to the linear maps $l_\infty(\Omega) \to \mathbb{R}$. Thus, what is proved in [24] is the existence of a universal constant $K < 45$ *with the property that if* $f : \mathcal{A} \to \mathbb{R}$ *is* Δ*-approximately additive, there is an additive function* $\mu : \mathcal{A} \to \mathbb{R}$ *with* $|f(A) - \mu(A)| \leq K \cdot \Delta$. In fact, they observe that it suffices to consider the case of finite algebras. The proof gets the additive map from the existence of a process called «concentrator». One of our purposes is to show that concentrators are actually quasi-linear non-linear maps.

Thus, in the way of understanding the proof, we became interested in the *methods* $F \to L(F)$ to obtain, in a K-space, linear maps at «almost optimal» finite distance. That is, the nature of the «almost optimal approximation map» $F \to L(F)$ such that, for some constant C, $\|F - L(F)\| \leq C\, \mathrm{dist}(F, X')$. We shall call to such map a *metric projection.* Which is the nature of the metric projection? Could it be even linear?

The interest in finding such linear method was fostered by the following attack: Let $f : \mathcal{A} \to \mathbb{R}$ be a 1-approximately additive function in a finite algebra $\mathcal{A}$. Suppose there exists a linear method $f \to m(f)$ to define, for some $r < 1$, a r-approximately additive map $m(f) : \mathcal{A} \to \mathbb{R}$ such that $|m(f)(A) - f(A)| \leq 1$. If so, we can iterate the method to obtain $m^2(f) : \mathcal{A} \to \mathbb{R}$ such that $|m^2(f)(A) - m(f)(A)| \leq r$ and $m^2(f)$ would be r^2-approximately additive; and so on. The sequence $(m^2(f))$ is contained in the compact subset of $\mathbb{R}^{\mathcal{A}}$:

$$\{g : |g(A)| \leq |f(A)| + (1 - r^{-1})\}.$$

Therefore, if $\mathcal{U}$ denotes a free ultrafilter on $\mathbb{N}$ then

$$L(f)(A) = \lim_{\mathcal{U}(n)} m_n(f)(A)$$

defines a linear map $L(f) : \mathcal{A} \to \mathbb{R}$ which verifies $|f(A) - L(f)(A)| \leq (1 - r^{-1})$. In the end, we would have obtained a linear metric projection $f \to L(f)$. Can we do this?

We do not want to spoil the forthcoming surprises, so we shall only say: no.

2. PRELIMINARIES

A quasi-norm on a (real or complex) vector space X is a nonnegative real-valued function $\|\cdot\|$ satisfying

i) $\|x\| = 0$ if and only if $x = 0$;

ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$;

iii) $\|x + y\| \leq K(\|x\| + \|y\|)$ for some constant K independent of $x, y \in X$.

A quasi-normed space is a vector space X together with a specified quasi-norm. On such a space one has a (vector) topology defined by the fundamental system of neighborhoods of 0 given by the multiples of the set $\{x \in X: \|x\| \leq 1\}$, called the unit ball of the quasi-norm. A complete quasi-normed space is called a *quasi-Banach* space. In the sequel, the word *operator* means linear continuous map. The algebraic dual X' of X is the space of linear, not necessarily continuous, maps; it shall also be denoted $\mathbf{L}(X, \mathbb{R})$, or simply $\mathbf{L}$. The subspace of X' formed by the linear continuous maps, the topological dual of X, shall be denoted X^*. An operator $X \to Y$ means always a linear continuous map. The space of homogeneous and bounded (i.e., such that the image of the unit ball is a bounded set) maps shall be denoted $\mathbf{B}(X, \mathbb{R})$, or simply $\mathbf{B}$. The term *bounded* map shall always mean homogeneous bounded map. Given two homogeneous maps A, B acting between the same spaces, their (eventually infinite) distance is defined as

$$\|A - B\| = \sup_{\|x\| \leq 1} \|Ax - Bx\|.$$

Exact sequences of (quasi) Banach spaces. For general information about exact sequences the reader can consult [15]. Information about categorical constructions in the (quasi) Banach space setting can be found in the monograph [9]. A diagram $0 \to Y \to X \to Z \to 0$ of quasi-Banach spaces and operators is said to be an *exact sequence* if the kernel of each arrow coincides with the image of the preceding. This means, by the open mapping theorem, that Y is (isomorphic to) a closed subspace of X and the corresponding quotient is (isomorphic to) Z. We shall also say that X is a *twisted sum of Y and Z* or an *extension of Y by Z*. Two exact sequences $0 \to Y \to X \to Z \to 0$ and $0 \to Y \to X_1 \to Z \to 0$ are said to be equivalent if there is an operator T making the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \downarrow T & & \| & & \\ 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \end{array}$$

commutative. The following standard result of algebra (see [15]) and the open mapping theorem imply that T must be an isomorphism.

The 3-lemma. *Assume that one has a commutative diagram of vector spaces and linear maps*

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \alpha\downarrow & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \to & Y_1 & \to & X_1 & \to & Z_1 & \to & 0 \end{array}$$

with exact rows. If α and γ are injective (resp. surjective) so is β.

An exact sequence $0 \to Y \to X \to Z \to 0$ is said to split if it is equivalent to the trivial sequence $0 \to Y \to Y \oplus Z \to Z \to 0$. This already implies that X is isomorphic to the direct sum $Y \oplus Z$.

Quasi-linear and 0-linear maps. The by now classical theory of Kalton and Peck [21] describes short exact sequences of quasi-Banach spaces in terms of the so-called *quasi-linear maps.* A map $F: Z \to Y$ acting between quasi-normed spaces is said to be quasi-linear if it is homogeneous and satisfies that for some constant K and all points x, y in Z one has

$$\|F(x + y) - F(x) - F(y)\| \leq K(\|x\| + \|y\|).$$

The smallest constant satisfying the inequality above is denoted $Q(F)$ and referred to as the quasi-linearity constant of the map F. We shall denote $\mathcal{Q}(X, \mathbb{R})$ the space of all quasi-linear maps $X \to \mathbb{R}$.

We shall say that a quasi-linear map is *trivial* when it can be written as the sum of a linear and a bounded map; or else, when it is at finite distance from a linear map. Two quasi-linear maps F and G (defined between the same spaces) are said to be equivalent if F-G is trivial. In this case we shall also say as in [2] that F is *a version* of G (or vice versa). Quasi-linear maps give rise to twisted sums: given a quasi-linear map $F: Z \to Y$ then it is possible to construct a twisted sum, which we shall denote by

$Y \oplus_F Z$, endowing the product space $Y \times Z$ with the quasi-norm

$$\|(y, z)\| = \|y - F(z)\| + \|z\|.$$

Clearly, the map $Y \to Y \oplus_F Z$ sending of y to $(y, 0)$ is an into isometry, and so Y can be thought as a subspace of $Y \oplus_F Z$; moreover, the corresponding quotient is isometric to Z. Conversely, an exact sequence $0 \to Y \to X \to Z \to 0$ comes defined by a quasi-linear map: pick a bounded selection B for the quotient map q (which exists by the open mapping theorem) and then a linear selection L; the difference $B - L$ is quasi-linear and takes values in Y since $q(B - L) = 0$. The two processes are one inverse of the other and, moreover, one has the following fundamental result of [21].

Proposition 2.1. *Two exact sequences* $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *and* $0 \to Y \to Y \oplus_G Z \to Z \to 0$ *are equivalent if and only F and G are equivalent. Therefore, an exact sequence* $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *is equivalent to the trivial exact sequence* $0 \to Y \to Y \oplus Z \to Z \to 0$ *if and only F is trivial (i.e., F is at finite distance from some linear map).*

The quasi-Banach space $Y \oplus_F Z$ constructed via a quasi-linear map F need not be locally convex, even when Y and Z are. A result of Dierolf [11] asserts that there exists a nonlocally convex twisted sum of Y and Z if and only if there exists a nonlocally convex twisted sum of $\mathbb{R}$ and Z. Hence, a Banach space is a K-space when every twisted sum with $\mathbb{R}$ is locally convex. It is however possible to obtain a simple characterization of when a *given* twisted sum of Y and Z is locally convex: the key is to give the characterization in terms of the quasi-linear map F and not in terms of the factor spaces.

Definition. A quasi-linear map $F: Z \to Y$ acting between quasi-normed spaces is said to be 0-*linear* if there is a constant K such that whenever $\{x_i\}$ is a finite set of elements of Z then

$$\|F(\textstyle\sum_{i=1}^{i=n} x_i) - \sum_{i=1}^{n} F(x_i)\| \le K \sum_{i=1}^{n} \|x_i\|.$$

The smallest constant satisfying the inequality above is denoted $Z(F)$ and referred to as the 0-linearity constant of the map F. The space of all 0-linear maps $X \to \mathbb{R}$ shall be denoted $Z(X, \mathbb{R})$. One has (see [2, 7, 9]).

Proposition 2.2. *A twisted sum of Banach spaces* $Y \oplus_F Z$ *is locally convex (being thus isomorphic to a Banach space) if and only if F is* 0*-linear.*

It is clear that 0-linear maps are quasi-linear. It is not true, however, that quasi-linear maps are 0-linear. Ribe [29] provided the simplest example of a quasi-linear not 0-linear map $R : l_1 \to \mathbb{R}$ given by

$$R(x) = \textstyle\sum_i x_i \log |x_i| - \sum_i x_i \log |\sum x_i|$$

(observe that the map is only defined on finitely supported sequences; however there exist extension theorems for quasi and 0-linear maps (see [21])). The quasi-linearity can be seen in [22] (actually $Q(R) = 2$) while the fact that R is not 0-linear is very simple to check: $R(e_n) = 0$ for all n while $R(\sum_{i=1}^{i=N} e_i) = -N \log N$; since $\sum_{i=1}^{i=N} \|e_i\| = N$, the estimate in the definition of 0-linear map is impossible.

It is moreover clear that a quasi-linear map F such that $\|F - L\| \le K$ for some linear map K necessarily is 0-linear and $Z(F) \le 2K$. Hence $Z(F) \le \text{dist}\,(F, \mathbf{L})$. In particular, Ribe's map R cannot be approximated by linear maps. As for the converse, one can see that using the Hahn-Banach theorem. Proposition 2.2 can be reformulated in terms of approximation by linear maps as follows (we shall give a direct proof for this result later):

Proposition 2.3. *A quasi-linear map* $X \to \mathbb{R}$ *is* 0*-linear if and only if it is at finite distance from a linear map.*

In this way we obtain that a Banach space X is a K-space if and only if every quasi-linear map $X \to \mathbb{R}$ is 0-linear.

The pull-back square. Let $A : U \to Z$ and $B : V \to Z$ be two arrows in a given category **C**. The *pull-back* of $\{A, B\}$ is an object Ξ in **C** and two arrows $u : \Xi \to U$ and $v : \Xi \to V$ such that $Au = Bv$; and such that given another object Γ in **C** for which there exist arrows $\alpha : \Gamma \to U$ and $\beta : \Gamma \to V$ verifying $A\alpha = B\beta$ then there exists a unique arrow $\gamma : \Gamma \to \Xi$ such that $\beta = v\gamma$ and $\alpha = u\gamma$. If one prefers the categorical language, the pull-back makes commutative the diagram

$$\begin{array}{ccc} U & \xrightarrow{A} & Z \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle B} \\ \Xi & \xrightarrow[v]{} & V \end{array}$$

and is universal with respect to this property.

In the category of quasi-Banach spaces and operators, as well as in the subcategory of Banach spaces pull-backs exist. If $A : U \to Z$ and $B : V \to Z$ are two operators, the pull-back of $\{A, B\}$ is the space $\Xi = \{(u, v) : Au = Bv\}$ endowed with the induced product topology together with the restrictions of the canonical projections of $U \oplus V$ onto, respectively, U and V. If $0 \to Y \to X \to Z \to 0$ is an exact sequence with quotient map q and $T: M \to Z$ is a surjective operator and Ξ denotes the pull-back of the couple $\{q, T\}$ then the diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \| & & \uparrow & & \uparrow{\scriptstyle T} & & \\ 0 & \to & Y & \to & \Xi & \to & M & \to & 0 \end{array}$$

is commutative with exact rows and columns.

3. LINEAR METRIC PROJECTIONS ON BANACH SPACES

As we have already seen, 0-linear maps on Banach spaces can be approximated by linear maps; thus, one has the decomposition

$$Z(X, \mathbb{R}) = \mathbf{B}(X, \mathbb{R}) + \mathbf{L}(X, \mathbb{R})$$

On a quasi-Banach K-space one even has

$$Q(X, \mathbb{R}) = \mathbf{B}(X, \mathbb{R}) + \mathbf{L}(X, \mathbb{R}).$$

Given a quasi-linear map F, let $D(F) = \text{dist}\big(F, \mathbf{L}(X, \mathbb{R})\big)$. Our main concern now is the nature of the map $F \to m(F)$ that associates to F an «almost optimal» selection, i.e. $m(F)$ is a linear map such that $\|F - m(F)\| \leq C\, D(F)$ (with C a prescribed finite constant). We have already seen that $Z(\cdot) \leq 2D(\cdot)$, hence it will be enough to study methods m such that $\|F - m(F)\| \leq C\, Z(F)$.

Our questions now are:

Question 1. *Do there exist Banach K-spaces in which the metric projection*

$$m : Q(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$$

is linear?

Question 2. *Do there exist quasi-Banach K-spaces in which the metric projection*

$$m : Q(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$$

is linear?

Observe that the hypothesis of being a K-space is necessary. Without it we can only ask:

Question 3. *Do there exist Banach spaces in which the metric projection*

$$m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$$

is linear?

We begin answering questions 1 and 3.

Proposition 3.1. *The metric projection* $m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ *is linear if and only if* X *is an* $\mathcal{L}_1$*-space.*

Proof. Let us consider first the case of a quasi-linear map $F : l_1^n \to \mathbb{R}$. Obviously $D(F)$ is finite and F is 0-linear. If (e_k) is the unit vector basis of l_1^n, we can define a linear map by $l(e_k) = F(e_k)$ (and linearly on the rest). We then have that for $x = \sum_k x_k e_k$ in l_1^n

$$|F(\textstyle\sum_k x_k e_k) - l(\sum_k x_k e_k)| \leq |F(\sum_k x_k e_k) - \sum_k x_k F(e_k)| \leq$$
$$\leq Z(F)\, \|\textstyle\sum_k x_k e_k\| = Z(F)\, \|x\|$$

and thus $\|F - l\| \leq Z(F)$. The correspondence $F \to m(F) = l$ is clearly linear.

We pass to an infinite dimensional $\mathcal{L}_1$-space X; let $F : X \to \mathbb{R}$ be a 0-linear map. Assume that $X = \cup X_\alpha$ where X_α is λ-isomorphic to l_1^α and X_α is λ-complemented in X. For each α, the map $F_\alpha = F_{|X_\alpha} : X \to \mathbb{R}$ admits a linear map $l_\alpha : X_\alpha \to \mathbb{R}$ such that $\|F_\alpha - l_\alpha\| \leq \lambda Z(F)$. Let L_α be an extension of l_α to the whole X obtained by setting $L_\alpha(y) = 0$ when y does not belong to X_α. Since for every x and eventually all α one has $|L_\alpha(x)| \leq \|F(x)\| + \lambda\, Z(F)$ it makes sense to define a linear map $L : X \to \mathbb{R}$ by

$$L(x) = \lim_{\mathcal{U}(\alpha)} L_\alpha(x)$$

where $\mathcal{U}$ is a free ultrafilter on index set (α) refining the Fréchet filter with respect to the natural ordering defined by the net (X_α). The application L is well defined and linear. One moreover has $\|F - L\| \leq \lambda Z(F)$ as follows from the following inequality choosing the index α carefully after ε:

$$|L(x) - F(x)| \leq |L(x) - L_\alpha(x)| + |L_\alpha(x) - F(x)| \leq \varepsilon + \lambda\, Z(F).$$

Finally, the procedure $F \to m(F) = L$ is linear.

We pass to the converse implication. Let X be a Banach space and assume the existence of a linear map $m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ such that $\|F - m(F)\| \leq C \cdot D(F)$.

Applying a uniform boundedness principle of Kalton [17] (the reader shall find a careful description of such principles in [3], there exists a constant C such that for every 0-linear map $D(F) \leq C\, Z(F)$.

Let now V be an ultrasummand; i.e., a Banach space complemented in its bidual. Let $G : X \to V$ be an arbitrary 0-linear map. We define a map $L : X \to V^{**}$ by

$$\langle L(x), v^* \rangle = \langle m(v^* \circ G), x \rangle,$$

which is linear since m is linear, and well defined since $L(x)$ is continuous:

$$\|L(x) = \sup \{\langle L(x), v^* \rangle : \|v^*\| \leq 1\} =$$
$$= \sup \{\langle m(v^* \circ G), x \rangle : \|v^*\| \leq 1\} =$$
$$= \sup \{\langle m(v^* \circ G) - v^* \circ G, x \rangle + \langle v^* \circ G, x \rangle : \|v^*\| \leq 1\} \leq$$
$$\leq \sup \{C\, D(v^* \circ G)\, \|x\| + \|v^*\|\, \|G(x)\| : \|v^*\| \leq 1\} \leq$$
$$\leq \sup \{C' Z(v^* \circ G)\, \|x\| + \|v^*\|\, \|G(x)\| : \|v^*\| \leq 1\} \leq$$
$$\leq \sup \{C' \|v^*\| Z(G)\, \|x\| + \|v^*\|\, \|G(x)\| : \|v^*\| \leq 1\} \leq$$
$$\leq C' Z(G)\, \|x\| + \|G(x)\|.$$

Since

$$\begin{aligned}|\langle G(x) - L(x), v^*\rangle| &= |\langle G(x), v^*\rangle - \langle m(v^* \circ G), x\rangle| = \\ &= |v^* \circ G(x) - m(v^* \circ G)(x)| \leq \\ &\leq 2C\, Z(v^* \circ G)\, \|x\| \leq \\ &\leq 2C\, \|v^*\|\, Z(G)\, \|x\|\end{aligned}$$

we get

$$\begin{aligned}\|G - L\| &= \sup_{\|x\| \leq 1} \|G(x) - L(x)\| = \\ &= \sup_{\|x\| \leq 1} \sup_{\|v^*\| \leq 1} |\langle G(x), v^*\rangle - \langle L(x), v^*\rangle| \leq \\ &\leq 2 \cdot C \cdot Z(G).\end{aligned}$$

To conclude we shall prove a result asserting that in the situation just described the space X has to be an $\mathcal{L}_1$-space. The if part is a result of Lindenstrauss [27] (although our proof shall be «considerably simpler») while, although the result is essentially known, we have no explicit reference for the only if part.

Proposition 3.2. *A Banach space Q is an $\mathcal{L}_1$-space if and only if for every ultrasummand Y every exact sequence $0 \to Y \to X \to Q \to 0$ splits.*

Proof. Assume that every exact sequence $0 \to Y \to W \to Q \to 0$ splits when Y *is* complemented in its bidual. We shall prove that $\boldsymbol{Q^*}$ is injective. For this, consider a exact sequence $0 \to \boldsymbol{Q^*} \to X \to Z \to 0$. One has

$$\begin{array}{ccccccccc} 0 & \to & Z^* & \to & X^* & \to & Q^{**} & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & Z^* & \to & P & \to & Q & \to & 0 \end{array}$$

where P is the pull-back of the quotient map $X^* \to Q^{**}$ and $Q \to Q^{**}$ is the canonical inclusion. Observe the diagram

$$\begin{array}{ccccccccc} 0 & \to & Q^* & \to & X & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q^{***} & \to & X^{**} & \to & Z^{**} & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & Q^* & \to & P^* & \to & Z^{**} & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & Q^* & \to & PB & \to & Z & \to & 0 \end{array}$$

where the second row is the bitraspose of the first row, and the second and third rows form the adjoint of the previous pull-back diagram. The third and fourth rows form a pull-back diagram with respect to the quotient map $P^* \to Z^{**}$ and the canonical inclusion $Z \to Z^{**}$.

The third row splits since it is transpose of the sequence $0 \to Z^* \to P \to Q \to 0$, which splits since Z^* is complemented in its bidual; thus, the fourth rows splits. But the first and fourth sequences are equivalent: since PB is the pull-back space of $P^* \to Z^{**}$ and $Z \to Z^{**}$, and we have arrows $X \to Z$ (quotient map in the first line) and $X \to P^*$ (vertical central line downwards) making a commutative square with the two previous arrows, there must exist an arrow $\alpha : X \to PB$ making the two triangles commutative. That makes the restriction $\alpha|Q^* = id$, and means that the upper and lower sequences are equivalent.

Now a proof for Lindenstrauss statement. Let Z be an $\mathcal{L}_1$-space and let $0 \to Y \to X \to Z \to 0$ be an exact sequence in which Y is an ultrasummand. Consider the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X & \to & Z & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Y^{**} & \to & X^{**} & \to & Z^{**} & \to & 0 \end{array}$$

Since Z^* is an injective space the dual sequence $0 \to Z^* \to X^* \to Y^* \to 0$ splits, and so does the bidual sequence; hence, Y^{**} is complemented in X^{**}, since Y *is* complemented in Y^{**}, it turns out that Y must be complemented in X and the original sequence splits. □

From all this we conclude:

Corollary 3.3. *Let X be a Banach space. It does not exist a linear metric projection*

$$\mathcal{Q}(X, \mathbb{R}) \to \mathbf{L}(X, \mathcal{R}).$$

Proof. Since, that would imply a linear metric projection $\mathcal{Z}(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ and, as we have seen, then X would be an $\mathcal{L}_1$-space. But $\mathcal{L}_1$-spaces are not K-spaces, and thus they admit quasi-linear maps that cannot be approximated by linear maps, which makes the existence of any selection method impossible. □

4. LINEAR METRIC PROJECTIONS ON QUASI-BANACH SPACES

Quasi-Banach spaces, however, conceal some surprises worth being uncovered. Let thus X be a quasi-Banach K-space. Assume moreover that it has trivial dual; i.e., $X^* = 0$ (here is where we need to have X not locally convex). The spaces $L_p(0, 1)$ with $0 < p < 1$ provide good examples of this situation.

Since X is a K-space, $\mathrm{Q}(X, \mathbb{R}) = \mathbf{B}(X, \mathbb{R}) + \mathbf{L}(X, \mathbb{R})$. Since X has trivial dual then $\mathbf{B}(X, \mathbb{R}) \cap \mathbf{L}(X, \mathbb{R}) = \{0\}$ (no map $X \to \mathbb{R}$ can be simultaneously linear and continuous). Therefore $\mathcal{Q}(X, \mathbb{R}) = \mathbf{B}(X, \mathbb{R}) \times \mathbf{L}(X, \mathbb{R})$. Let us show now that the canonical projection onto $\mathbf{L}(X, \mathbb{R})$ is, in addition to linear, a metric projection.

To this end, let us recall that given a quasi-Banach space X one can consider two semi-metrics (they are not Hausdorff) on $\mathcal{Q}(X, \mathbb{R}) : Q(\cdot)$ and $d(\cdot) = \mathrm{dist}(\cdot, \mathbf{L})$. Let us observe that they are equivalent: the uniform boundedness principle mentioned earlier shows that the two induced norms are equivalent on $\mathcal{Q}(X, \mathbb{R})/\mathbf{L}$; now, $\mathbf{L}$ is the kernel of the two seminorms, and thus they are also equivalent.

In the present situation $Q(X, \mathbb{R}) = \mathbf{B}(X, \mathbb{R}) \times \mathbf{L}(X, \mathbb{R})$ they adopt the form $Q(b, l) = Q(b)$; and $d(b, l) = \text{dist}\,(b, \mathbf{L})$. The application

$$n(b, l) = \|b\|$$

defines a complete (since the space $\mathbf{B}(X, \mathbb{R})$ is complete in this norm) seminorm on $Q(X, \mathbb{R})$; since $n \geq d$, it turns out to be also equivalent to $d(\cdot)$ and $Q(\cdot)$. But the canonical projection

$$m(b, l) = l$$

is a metric projection for n; i.e., that $n\big(F - m(F)\big) \leq C\, Q(F)$:

$$n\big(b + l - m(b, l)\big) = \|b\|. \qquad \square$$

5. LINEAR METRIC PROJECTIONS FOR GER-LINEAR MAPS

As we have already seen, given an arbitrary Banach space, no *linear* method $F \longrightarrow m(F)$ is able to assign to each quasi-linear map F a linear map $m(F)$ at a prefixed distance C. Could such linear method be obtained if one restricts the attention to smaller subclasses of quasi-linear maps? For instance, for 0-linear maps such linear method exists in $\mathcal{L}_1$-spaces.

Until now we have only considered two classes: the class $\mathcal{Q}$ of quasi-linear maps and the class $\mathcal{Z}$, of 0-linear maps. There exist other interesting classes worth consideration. One of them was isolated by Lima and Yost in [25]: the class $\mathcal{P}$ of pseudo-linear maps, that is, quasi-linear maps Ω satisfying

$$\|\Omega(x + y) - \Omega(x) - \Omega(y)\| \leq \|x\| + \|y\| - \|x + y\|.$$

The appendix 1.9 in [9] contains a rather complete survey about these maps. Another class introduced and studied in [5] (see also [14]) is formed by the Ger-linear maps. A quasi-linear map $F : X \longrightarrow Y$ is said to be Ger-linear if

$$\|F(x + y) - F(x) - F(y)\| \leq C\|x + y\|$$

for some constant $C > 0$ and all $x, y \in X$. The infimum of those constants C verifying the previous inequality is called the Ger-linearity constant of G and denoted $G(F)$. The space of all Ger-linear maps $X \longrightarrow Y$ shall be denoted $\mathcal{G}(X, Y)$. A simple induction argument shows that a Ger-linear map is 0-linear and $Z(\cdot) \leq G(\cdot)$.

The interesting feature of Ger-linear maps is their connection with classical problems about the existence of Lipschitz projections on Banach spaces. More precisely (see [5])

Proposition 5.1. *Are exact sequence of Banach spaces $0 \longrightarrow Y \longrightarrow X \longrightarrow Z \longrightarrow 0$ is defined by a Ger-linear map $G : Z \longrightarrow Y$ if and only if there exists a Lipschitz projection from X onto Y. Moreover, that happens if and only if the metric projection $\rho : Y \oplus_G Z \longrightarrow Y$ given by $\rho(y, z) = y - G(z)$ is Lipschitz.*

It is still an open problem to know if there exist nontrivial pseudo-linear maps. However, nontrivial Ger-linear maps do exist: it is not hard to verify that the Aharoni-Lindenstrauss (nontrivial) sequence $0 \longrightarrow C[0, 1] \longrightarrow D \longrightarrow c_0 \longrightarrow 0$ (see [1]) comes defined by a Ger-linear map. The interesting point for us now is that, as it was shown in [5], Ger-linear or pseudo-linear maps from a Banach space into an ultrasummand are trivial. We show now that the metric projection for Ger-linear maps is linear.

As proposition 5.1 suggests, and almost proves, and contrarily to intuition, not all trivial maps are Ger-linear maps. Indeed, if a Ger-linear map G is trivial then not only it can be decomposed $G = B + L$ as a sum of a bounded homogeneous plus a linear map; in this case the bounded map has to be Lipschitz (to make Lipschitz the map ρ).

So, when all Ger-linear maps are trivial we shall write $\mathcal{G} = \mathbf{B}_1 + \mathbf{L}$ to indicate that the bounded map is Lipschitz. The example of the Aharoni-Lindenstrauss construction shows that the hypothesis «Y is an ultrasummand» in the next proposition is not superfluous.

Proposition 5.2. *Let X be a quasi-Banach space and let Y be a quasi-Banach ultrasummand. Then all Ger-linear maps $X \longrightarrow Y$ are trivial and, moreover, there exist a linear metric projection*

$$m : \mathcal{G}(X, Y) \longrightarrow \mathbf{L}(X, Y)$$

(of course, the same linear method would work for pseudo-linear maps).

Proof. Let μ be a Banach limit (i.e., an invariant mean) in the commutative group $(X, +)$ and let $\pi : Y^{**} \longrightarrow Y$ be a projection. We define

$$m(G)(x) = \pi\big(\text{weak*-lim}_{\mu(y)}\, G(x + y) - G(y)\big).$$

Observing that the definition of Ger-linear map could have also been (how could G recognize who is x, who is y and who is $x + y$?)

$$\|G(x + y) - G(x) - G(y)\| \leq C\,\|x\|$$

it follows that $||G(x + y) - G(y)|| \leq C||x|| + ||G(x)||$, and thus $\{G(x + y) - G(y)\}_{y \in X}$ lies in a weak*-compact set and using a Banach limit makes sense. Since

$$m(G)(x + z) = \text{weak*-lim}_{\mu(y)} G(x + z + y) - G(y) =$$
$$= \text{weak*-lim}_{\mu(y)} G(x + z + y) -$$
$$- G(z + y) + G(z + y) - G(y) =$$
$$= \text{weak*-lim}_{\mu(z+y)} G(x + z + y) - G(z + y) +$$
$$+ \text{weak*-lim}_{\mu(y)} G(z + y) - G(y) = m(G)(x) + m(G)(z)$$

we have the linearity of $m(G)$. Moreover, for every $\varepsilon > 0$ one can choose y^* so that

$$||m(G)(x) - G(x)|| \leq |y^*(m(G)(x) - G(x))| + \varepsilon \leq$$
$$\leq 2\varepsilon + |y^*(G(x + y) - G(y)) - G(x))| \leq$$
$$\leq 2\varepsilon + ||G(x + y) - G(y) - G(x)|| \leq$$
$$\leq 2\varepsilon + C||x||.$$

We now show that m is a linear metric projection. There is little doubt that it is linear. To show that it is a metric projection let us show that $G(\cdot)$ is proportional to $\text{dist}(\cdot, \mathbf{L}(X, Y))$. We shall shorten for the rest of this proof $\mathbf{L}(X, Y)$ to simply $\mathbf{L}$.

Proposition 5.3. *Let Y and Z be two Banach spaces. Assume that all Ger-linear maps $Z \to Y$ are trivial. Then there is a constant ρ such that for every Ger-linear map $F: Z \to Y$, one has $dist(F, \mathbf{L}) \leq \rho\, G(F)$.*

Proof. Consider the following two norms on $\mathbf{B}_1 + \mathbf{L}/\mathbf{L}$. The first one is the quotient metric $D(\cdot) = \text{dist}(\cdot, \mathbf{L})$, and the other is $G(\cdot)$. One has □

Lemma 5.4. $(\mathbf{B}_1 + \mathbf{L}/\mathbf{L}, D)$ *is a Banach space.*

Proof. Easy, since $\mathbf{B}_1 + \mathbf{L}/\mathbf{L} = \mathbf{B}_1 + \mathbf{L}/\mathbf{L} = \mathbf{B}_1/\mathbf{B}_1 \cap \mathbf{L}$ and $(\mathbf{B}_1, d)$ is complete. □

Lemma 5.5. $(\mathcal{G}(Z, Y)/\mathbf{L}, G)$ *is a Banach space.*

Proof. Let $([G_n])$ be a G-Cauchy sequence. Fix a normalized Hamel basis $(e_\gamma)_\gamma$ for Z and observe that if $H: Z \to Y$ is a Ger-linear map then there exists a unique representative F of $[H]$ vanishing on all the elements of the basis; take

$$F(\Sigma_\gamma \lambda_\gamma e_\gamma) = H(\Sigma_\gamma \lambda_\gamma e_\gamma) - \Sigma_\gamma \lambda_\gamma H(e_\gamma).$$

From now on, F_n shall be the representative of $[H_n]$ vanishing on the basis. The sequence (F_n) is pointwise convergent because if $z = \Sigma \lambda_\gamma e_\gamma$ then

$$||(F_n - F_m)(z)|| \leq Z(F_n - F_m)\Sigma|\lambda_\gamma| \leq G(F_n - F_m)\Sigma|\lambda_\gamma|.$$

Let F be its pointwise limit,

$$F(z) = \lim F_n(z).$$

We show that $[F]$ is the G-limit of $([F_n]) = ([H_n])$. Let (z_j) be a finite set of points such that $\Sigma_i z_i = 0$, and let $\varepsilon > 0$. Choose indices $n(j)$ so that y $||(F - F_{n(j)})(z_j)|| < 2^{-j}\varepsilon$. One has:

$$||\Sigma_j(F - F_n)(z_j)|| \leq ||\Sigma_j(F - F_{n(j)})(z_j) + F_{n(j)})(z_j) - F_n(z_j)|| \leq$$
$$\leq \Sigma_j ||(F - F_{n(j)})(z_j)|| + G(F_{n(j)} - F_n)\Sigma_j ||z_j|| \leq$$
$$\leq \varepsilon + G(F_{n(j)} - F_n)\Sigma_j ||z_j||$$

which is everything one needs since the sequence $([F_n])$ was G-Cauchy. From that it also follows that F is Ger-linear since $G(F) \leq G(F - F_n) + G(F_n)$. □

End of the proof. Since $G(\cdot) \leq D(\cdot)$ on $\mathcal{G} = \mathrm{B}_1 + \mathbf{L}$ then, the norms G and D are comparable on $\mathcal{G}/\mathbf{L} = \mathbf{B}_1 + \mathbf{L}/\mathbf{L}$. The open mapping theorem ensures that G and D are equivalent. □

6. SUB-LINEAR METHODS FOR 0-LINEAR MAPS

What has happened recently raises again the doubt: what occurs with 0-linear maps that no linear method is available? The answer could be that Ger-linear maps seem to be nicely coupled with Banach limits, while the class of 0-linear maps (whose definition involves many decompositions into a finite number of points) does not seem to be suitable to match with a *single* linear method. As a further evidence it is the fact that 0-linear maps $X \to Y$ are not automatically trivial when Y is an ultrasummand (even reflexive! recall the existence ofnontrivial sequences, say, $0 \to l_2 \to l_\infty \to l_\infty/l_2 \to 0$). Following this line, we show now that there is a method $F \to m(F)$ for obtaining almost optimal linear maps which can be decomposed in only two methods, one of them linear and the other sub-linear. This will show that 0-linear maps, if not as polite as Ger-linear maps, are definitely not totally disastrous.

Proposition 6.1. *Let X be a quasi-Banach space. There is a metric projection $m: \mathcal{Z}(X, \mathbb{R}) \to \mathbf{L}$ that can be decomposed as*

$$m = \lambda\, m_1$$

where m_1 is sub-linear and λ is linear.

Proof. It is not hard to see that the preceding method (using an invariant mean to get a linear map) not only works whit Ger-linear maps; it actually works with sub-linear maps S such that $S(\lambda x) = \lambda\, S(x)$ for positive λ (that we shall call +-homogeneous). Let $\mathcal{S}(X, \mathbb{R})$ be the class of all sublinear +-homogeneous functions $X \to \mathbb{R}$. If

$S \in \mathcal{S}(X, \mathbb{R})$ then $|S(x + y) - S(y)| \leq \max\{|S(x)|, |S(-x)|\}$ and the method

$$\lambda(S)(x) = \lim_{\mu(y)} S(x + y) - S(y)$$

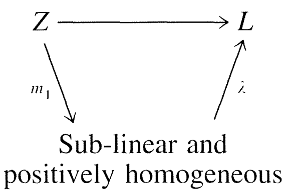
still provides a linear map.

Now, let F be a 0-linear map with constant $Z(F)$; then if we define

$$m_1(F)(x) = \inf\left\{\sum_{i=1}^{n} F(x_i) + Z(F) \sum_{i=1}^{n} \|x_i\| : x = \sum_{i=1}^{n} x_i\right\}$$

what we get is a sub-linear and +-homogeneous map $m_1(F)$ satisfying $\|F - m_1(F)\| \leq Z(F)$. To prove this last assertion, note that $m_1(x) \leq F(x) + Z(F)\|x\|$ while for no matter which decomposition $x = \sum x_i$ we have, by the definition of 0-linear map $F(x) \leq \sum F(x_i) + Z(F)\|x\|$.

So, the composition method



yields a «sub-linear» metric projection.

Let us show now that, against what we could guess, this situation is perfectly reasonable.

7. THE METRIC PROJECTION IS A QUASI-LINEAR MAP

We only have to enlarge our working category. Let **Met** be the category of vector spaces endowed with a metric, and linear lipschitz maps as arrows. Our key examples are $(\mathbf{L}, d)$, $(\mathbf{B} + \mathbf{L}, d)$ and $((\mathbf{B} + \mathbf{L})/\mathbf{L}, D) = ((\mathbf{B} + \mathbf{L})/\mathbf{L}, Z(\cdot))$, where $d(A, B) = \|A - B\|$ and $D(\cdot) = dist(\cdot, \mathbf{L})$ is the induced metric.

Let $q : \mathbf{B} + \mathbf{L} \to (\mathbf{B} + \mathbf{L})/\mathbf{L}$ be the quotient map, and let $s : (\mathbf{B} + \mathbf{L})/\mathbf{L} \to \mathbf{B} + \mathbf{L}$ be a linear selection for q. We define the map $G : (\mathbf{B} + \mathbf{L})/\mathbf{L} \to \mathbf{L}$ by means of

$$G(x + \mathbf{L}) = x - m(x) - s(x + \mathbf{L}).$$

Lemma 7.1. *The map $G : [(\mathbf{B} + \mathbf{L})/\mathbf{L}, D] \to [\mathbf{L}, d]$ is quasi-linear.*

Proof. Keep in mind that m_1 satisfies $m_1(b + l) = m_1(b) + l$, while λ is linear. This makes G well defined since $x - s(x + \mathbf{L}) \in \mathbf{L}$ and $m(x) \in \mathbf{L}$; and, moreover, if $x - y = l \in \mathbf{L}$ then

$$\begin{aligned} G(x + L) &= x - m(x) - s(x + L) = \\ &= y + l - m(y + l) - s(y + L) = \\ &= y + l - m(y) - l - s(y + L) = \\ &= G(y + L). \end{aligned}$$

The quasi-linearity of G means that:

$$\begin{aligned} &\operatorname{dist}(G(x + y + L), G(x + L) + G(y + L)) = \\ &= \|x + y - m(x + y) - x + m(x) - y + m(y)\| \leq \\ &\leq 2(1 + \varepsilon)(\|x + L\| + \|y + L\|). \end{aligned}$$ □

One should not be surprised. After all, the map G has been constructed in the standard way for a quasi-linear map:

$$G(x + L) = \underbrace{x - m(x)}_{\substack{\text{bounded selection}\\ \text{for } q}} - \underbrace{s(x + L)}_{\substack{\text{linear selection}\\ \text{for } q}}$$

and since the two selections are defined $(\mathbf{B} + \mathbf{L})/\mathbf{L} \to \mathbf{B} + \mathbf{L}$ and the kernel of q is precisely $\mathbf{L}$ it is not strange to get:

Lemma 7.2. *The quasi-linear map G defines, in the category **Met**, the exact sequence*

$$0 \to \mathbf{L} \to \mathbf{B} + \mathbf{L} \to (\mathbf{B} + \mathbf{L})/\mathbf{L} \to 0.$$

Proof. To check that, we construct the exact sequence

$$0 \to \mathbf{L} \to \mathbf{L} \oplus_G (\mathbf{B} + \mathbf{L})/\mathbf{L} \to (\mathbf{B} + \mathbf{L})/\mathbf{L} \to 0$$

in the standard way: the metric in the twisted sum space is

$$\begin{aligned} \rho((y, z), (y', z')) &= |y - y', z - z'\|_G = \\ &= \|y - y' - G(z - z')\| + \|z - z'\|; \end{aligned}$$

and show that the two sequences are equivalent: the map $T(x) = (x - s\, q(x), q(x))$ is obviously linear, makes the diagram

$$\begin{array}{ccccc} 0 \to L \to & B + L & \to (B + L)/L \to 0 \\ \| & \downarrow T & \| \\ 0 \to L \to & L \otimes_G (B + L)/L & \to (B + L)/L \to 0 \end{array}$$

commutative and is lipschitz:

$$\begin{aligned} &\rho((x - s\, q(x), q(x)), (y - s\, q(y), q(y))) = \\ &= \|(x - s\, q(x) - y + s\, q(y), q(x) - q(y)\| = \\ &= \|(x - y - s\, q(x - y), q(x - y)\| = \\ &= \|x - y - s\, q(x - y) - G\, q(x - y)\| + \|q(x - y)\| = \\ &= \|x - y - m(q(x - y)) - (x - y)\| + \|q(x - y)\| \leq \\ &\leq 3\|x - y\|. \end{aligned}$$ □

We are ready for a nice result. It is only a little of glittering make-up to write the algebraic dual X' instead of $\mathbf{L}(X, \mathbb{R})$ and the topological dual X^* instead of $\mathbf{B}(X, \mathbb{R}) \cap \mathbf{L}(X, \mathbb{R})$, and $\mathbf{B}/\mathbf{B} \cap \mathbf{L}$ instead of $(\mathbf{B} + \mathbf{L})/\mathbf{L}$.

Proposition 7.3. *Let X be a Banach space. The following are equivalent:*

i) There exists a linear metric projection $m : Z(X, \mathbb{R}) \to X'$

ii) $Z(X, \mathbb{R}) = X' \oplus \mathbf{B}(X, \mathbb{R})/X^*$.

iii) X' *is complemented in* $Z(X, \mathbb{R})$.

Proof. As for the proof, just observe that if m were linear, G would be trivial and the sequence

$$0 \to X' \to Z(X, \mathbb{R}) \to \mathbf{B}/X^* \to 0$$

would split. And conversely, if this sequence splits then X' is complemented in $Z(X, \mathbb{R})$; equivalently, there exists a linear metric projection $m : Z(X, \mathbb{R}) \to X'$. □

And also:

Proposition 7.4. *Let X be a quasi-Banach K-space. The following are equivalent:*

i) There exists a linear metric projection $m : \mathcal{Q}(X, \mathbb{R}) \to X'$.

ii) $\mathcal{Q}(X, \mathbb{R}) = X' \oplus \mathbf{B}(X, \mathbb{R})/X^*$.

iii) X' *is complemented in* $\mathcal{Q}(X, \mathbb{R})$.

We have seen so far two instances of this situation: the conditions in proposition 7.3 are equivalent to the fact that X is an $\mathcal{L}_1$-space; on the other hand, we know that conditions in proposition 7.4 hold when X is a K-space with trivial dual. Let us give a unifying theorem.

Recall that the Banach envelope of a quasi-Banach space $co(X)$ is defined as the closure in X^{**} of the canonical image of $X \to X^{**}$ under the map $\delta(x)(x^*) = x^*(x)$. It has the universal property that every operator $\tau: X \to \mathbb{R}$ admits an extension $T: co(X) \to \mathbb{R}$ such that $T\delta = \tau$. In this way we arrive to the central result of the paper.

Theorem 7.5. *Let X be a quasi-Banach space. Then there exist a linear metric projection* $m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ *if and only if* $co(X)$ *is an* $\mathcal{L}_1$*-space.*

Proof. The proof requires the duality techniques developed in [2] for Banach spaces and extended in [5] to quasi-normed groups. Precisely, that given a 0-additive map $f: G \to \mathbb{R}$ on a quasi-normed group there exists a 0-linear map $F: co(G) \to \mathbb{R}$ such that $F\delta$ is a version of f.

Now, if here exists a linear metric selection $m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ then the same proof of proposition 3.1 shows that every 0-linear map $X \to V$, where V is an ultrasummand, is trivial. By the result mentioned above, every 0-linear map $co(X) \to V$ is trivial, and thus, by the characterization 3.2, $co(X)$ is an $\mathcal{L}_1$-space.

Conversely, assume that $co(X)$ is an $\mathcal{L}_1$-space. The existence of a linear metric selection $m : Z(X, \mathbb{R}) \to \mathbf{L}(X, \mathbb{R})$ is equivalent to the splitting of the exact sequence

$$0 \to X' \to Z(X, \mathbb{R}) \to (\mathbf{B} + \mathbf{L})/\mathbf{L} \to 0;$$

hence, equivalent to the existence of a linear Lipschitz selection $s: (\mathbf{B} + \mathbf{L})/\mathbf{L} \to Z$. Since $(\mathbf{B} + \mathbf{L})/\mathbf{L} = \mathbf{B}/\mathbf{B} \cap \mathbf{L}$, we are asking about the existence of a linear Lipschitz selection $\mathbf{B}/X^* \to \mathbf{B} + \mathbf{L}$. But since $X^* = co(X)^*$ and $co(X)$ is an $\mathcal{L}_1$-space, X^* is injective. So, the sequence $0 \to X^* \to \mathbf{B} \to \mathbf{B}/X^* \to 0$ splits. A look at the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \to & B + L & \to & (B + L)/L & \to & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \to & X^* & \to & B & \to & B/X^* & \to & 0 \end{array}$$

should convince us that when the lower sequence splits so does the upper sequence. □

This result includes the previous cases: if X is itself a Banach space then $X = co(X)$. If X is a quasi-Banach with trivial dual then $co(X) = 0$, which is certainly an $\mathcal{L}_1$-space.

Things could be pushed further making homogeneity disappear and moving to quasi-Banach groups. The reader is referred to [5] for an introduction, reference and full development of the theory of quasi-additive maps on controlled semigroups. With essentially (except for a tricky point of the theory of groups: that 0-additive maps are not automatically close to an additive map) the same proof as before one gets.

Proposition 7.6. *Let* (G, ρ) *be a quasi-normed group such that every* 0*-additive map* $(G, \rho) \to \mathbb{R}$ *is asymptotically additive. The following are equivalent:*

a) There exist an additive metric projection $Z \to L$.

b) $co(G)$ *is an* $\mathcal{L}_1$*-space.*

8. THE UNIVERSAL *coz*(*X*) SPACE

The Banach envelope $co(X)$ of a quasi-Banach space is an universal object characterized by the following property: every operator $T: X \to Y$ into a Banach space factorizes through the operator $X \to co(X)$. Does there exists a similar object for 0-linear maps? The answer is yes and this new object provides a deep insight into the problem of finding a linear metric projection.

Proposition 8.1. *There exists a Banach space $coz(X)$ and a 0-linear map $\delta : X \to coz(X)$ with the property that for every 0-linear map $F : X \to \mathbb{R}$ there exists a linear continuous map $\pi_F : coz(X) \to \mathbb{R}$ such that $\pi_F \delta = F$.*

Proof. Let $coz(X) = [Z(X, \mathbb{R}), Z(\cdot)]^*$. The space $[Z, (X, \mathbb{R}), Z(\cdot)]$ is a semi-normed space (see [4] for a related construction yielding a semi-Banach space) space. The operator $\delta: X \to coz(X)$ is «essentially» obviously defined by $\delta(x)(F) = F(x)$. The reader may observe that $\delta(x) \in Z(X, \mathbb{R})'$ and might not be continuous. It is not difficult to define a linear map $L: X \to Z(X, \mathbb{R})'$ such that, for every $x \in X$, $\delta(x) - L(x) \in [Z(X, \mathbb{R}), Z(\cdot)]^*$: just consider a Hamel basis (x_γ) of norm one vectors of X and define $L(x_\gamma) = \delta(x_\gamma)$. It is clear now that

$$|F(\sum \lambda_\gamma x_\gamma) - \sum \lambda_\gamma F(x_\gamma)| \leq Z(F) \sum |\lambda_\gamma|$$

and thus $\|\delta(x) - L(x)\| \leq \sum |\lambda_\gamma|$. The presence of L does not modifies the 0-linear character of δ, which should be self-evident. Finally, if $F: X \to \mathbb{R}$ is a 0-linear map then since $[Z(X, \mathbb{R}), Z(\cdot)]^*$ is a vector subspace of $\mathbb{R}^{Z(X, \mathbb{R})}$, then π_F is the restriction to $[Z(X, \mathbb{R}), Z(\cdot)]^*$ of the projection onto the F-coordinate. The linearity and continuity of such map are obvious. □

What is interesting for us now is the following property:

Proposition 8.2. *There exists a linear metric selection $m : Z(X, \mathbb{R}) \to X'$ if and only if δ can be approximated by a linear map.*

Proof. It $S: X \to coz(X)$ is a linear map such that $\|\delta - S\| \leq M < +\infty$ then $\pi_F L: X \to \mathbb{R}$ is a linear map such that $\|F - \pi_F S\| \leq \|\pi_F\| \, MZ(F)$, as we show now:

$$|Fx - \pi_F Sx| = |\delta_x(F) - \pi_F S(x)| =$$
$$= |\pi_F(\delta_x - S)| \leq$$
$$\leq \|\pi_F\| \, \|\delta - S\| \, \|x\|.$$

On the other hand, there is little doubt that the process $F \to \pi_F L$ is linear.

Conversely, if there exists a linear metric selection $F \to m(F)$ then, as we have already seen, $co(X)$ is an $\mathcal{L}_1$-space, in which case every 0-linear map $co(X) \to V$ taking values in an ultrasummand is trivial (reasoning as in 3.1). By the duality results cited at the beginning of the proof of theorem 7.5, every 0-linear map from X into an ultrasummand is trivial. Since $coz(X)$ is a dual space, it is complemented in its bidual, hence it is an ultrasummand and thus δ is trivial. □

9. APPENDIX: A TWIST OF THE SCREW

In the previous sections we have worked with the sequence

$$0 \to \mathbf{L} \to \mathbf{B} + \mathbf{L} \to (\mathbf{B} + \mathbf{L})/\mathbf{L} \to 0$$

of metric spaces under the metric $\|\cdot\|$. We could have also considered the same sequence under the semi-metric $Q(\cdot)$; it is not Hausdorff because the linear maps form the closure of 0. To be a K-space means that the norms induced on $(\mathbf{B} + \mathbf{L})/\mathbf{L}$ by $\|\cdot\|$ and $Q(\cdot)$ coincide. Let us consider now the situation on finite dimensional spaces to recover the meaning of the Kalton-Roberts theorem and to put in perspective the results proved in the paper.

Let Ω be a finite set. Let $l_\infty(\mathcal{P}(\Omega))$ be the space of all (all = bounded) maps $\mathcal{P}(\Omega) \to \mathbb{R}$. The subspace of all additive maps is precisely $l_1(\Omega)$, and the embedding is

$$\begin{array}{ccl} l_1(\Omega) & \to & l_\infty(\mathcal{P}(\Omega)) \\ \mu & \to & \bar{\mu} \quad : \bar{\mu}(A) = \sum_{i \in A} \mu(i). \end{array}$$

Thus, one has the exact sequence (of vector spaces)

$$0 \to l_1(\Omega) \to l_\infty(\mathcal{P}(\Omega)) \to l_\infty(\mathcal{P}(\Omega)) / l_1(\Omega) \to 0$$

Consider now the sequence in the semi-norm

$$Q(\mu) = \sup \{|\mu(A \cup B) - \mu(A) - \mu(B)| : A, B \text{ disjoint}\}.$$

One has only an exact sequence of semi-Banach spaces since $l_1(\Omega) = \overline{\{0\}}^{Q(\cdot)}$; however, the quotient $l_\infty(\mathcal{P}(\Omega)) / l_1(\Omega)$ is a certain finite-dimensional Banach space.

However, if the sequence is considered in the usual $\|\cdot\|_\infty$ norm, the embedding of $l_1(\Omega)$ into $l_\infty(\mathcal{P}(\Omega))$ is nothing different from a Rademacher-like embedding of $l_1(n)$ into $l_\infty(2^n)$ (in fact, if $|\Omega| = n$ then $|\mathcal{P}(\Omega)| = 2^n$). This embedding is not so accurate as to be isometric since one can only obtain $2^{-1} \|\mu\|_1 \leq \|\bar{\mu}\| \leq \|\mu\|_1$. In this way, the sequence is just the exact sequence of Banach spaces

$$0 \to l_1(n) \to l_\infty(2^n) \to l_\infty(2^n) / l_1(n) \to 0$$

for which one is perfectly able to prove two things: that it splits (like all sequences with finite dimensional spaces do), and that it does with projections having norms tending to infinity (since l_1 is not complemented in an $\mathcal{L}_\infty$-space).

In this context, the Kalton-Roberts theorem says that the Banach spaces $(l_\infty(2^n) / l_1(n), \|\cdot\|_\infty)$ and $(l_\infty(2^n)) / l_1(n), Q(\cdot))$ are 90-isomorphic independently on n. While our theorem about the nonexistence of a linear metric projection method «essentially» means that l_1 is not complemented in $l_\infty(2^{\mathbb{N}})$.

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