Rev.R.Acad.Cienc.Exact.Fis.Nat. (Esp)
Vol. 94, N.º 4, pp 467-472, 2000
Monográfico: Perspectivas en Análisis Matemático

SOME APPLICATIONS OF PROJECTIVE TENSOR PRODUCTS TO HOLOMORPHY

JOSÉ M. ANSEMIL and SOCORRO PONTE

Departamento de Análisis Matemático, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid (Spain)

Presentado por F. Bombal.

ABSTRACT

In this paper we survey recent applications of the isomorphism between the spaces of all n-homogeneous continuous polynomials and the dual of the space of all n-symmetric tensors on the space endowed with the projective topology. Some of those applications have influenced the study of the space of all holomorphic functions on a balanced open subset of a complex locally convex space.

1. INTRODUCTION

Let E be a locally convex space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and for $n \in \mathbb{N}$ let $\mathcal{P}(^nE)$ be the space of all n-homogeneous continuous polynomials on E with values in $\mathbb{K}$. We recall that the elements in $\mathcal{P}(^nE)$ are the mappings $P : E \to \mathbb{K}$ such that there is an n-linear continuous mapping A from $E \times \cdots \times E$ into $\mathbb{K}$ that satisfies $P(x) = A(x, \ldots, x)$ for every $x \in E$.

The spaces $\mathcal{P}(^nE)$ are fundamental in the study of the spaces of holomorphic functions on the complex locally convex space E. The main reason is that for a complex locally convex space E and a balanced open subset U of E, the sequence $\left(\mathcal{P}(^nE)\right)_{n=0}^{\infty}$ is a Schauder decomposition for the space $\mathcal{H}(U)$ of all holomorphic functions on U endowed with any of its natural topologies [22]. As Ryan pointed out in his thesis [38], the space $\mathcal{P}(^nE)$ is algebraically isomorphic to the topological dual of the completion of the n-fold symmetric projective tensor product of E, noted by $\hat{\otimes}^n_{s,\pi}E$. That is

$$\mathcal{P}(^nE) \cong (\hat{\otimes}^n_{s,\pi}E)'. \qquad (*)$$

The topology we are considering on $\hat{\otimes}^n_s E$ is just the restriction to that space of the classical projective topology on $\hat{\otimes}^n E$. A natural description of this topology using only symmetric tensors can be given [26], [28].

This isomorphism has been used recently by different authors to obtain results on spaces of polynomials, some of which pass to spaces of holomorphic functions. Our aim here is to survey recent applications of the above isomorphism. These applications are related with topological properties $\mathcal{P}(^nE)$, so let us recall the usual notation for the different of natural topologies on $\mathcal{P}(^nE)$:

τ_0 will denote the compact open topology defined by the seminorms

$$P \in \mathcal{P}(^nE) \mapsto \sup_{x \in K} |P(x)|$$

when K ranges over the family of all compact subsets of E. τ_0 is the topology of uniform convergence on the compact subsets of E.

τ_b will denote the topology of uniform convergence on the bounded subsets of E. It is defined by the family of seminorms

$$P \in \mathcal{P}(^nE) \mapsto \sup_{x \in B} |P(x)|$$

when B ranges over the family of all bounded subsets of E.

β will stand for the strong topology on $\mathcal{P}(^nE)$ as the dual of $\hat{\otimes}^n_{s,\pi}E$. Note that if B is a bounded subset of E, then $\otimes^n_s B := \{\otimes^n x := x \otimes \cdots \otimes x;\ x \in B\}$ is a bounded subset of $\hat{\otimes}^n_{s,\pi}E$ and

$$\sup_{x \in B} |P(x)| = \sup_{x \in B} |\dot{P}(\otimes^n x)| = \sup_{z \in \otimes^n_s B} |\dot{P}(z)|.$$

where $\dot{P}$ denotes the linearization of P through the isomorphism (*).

So, $\tau_b \leq \beta$. But since in general not every bounded subset $\dot{B}$ in $\hat{\otimes}^n_{s,\pi}E$ is included in the closed convex hull of a set of the form $\otimes^n_s B$, with B bounded in E, we have in general (see Section 2 below), that $\tau_b < \beta$ on $\mathcal{P}(^nE)$. In fact, $\tau_b = \beta$ on $\mathcal{P}(^nE)$ if and only if E has the so called $(BB)_{n,s}$ property: E satisfies the $(BB)_{n,s}$ property for $n = 2, 3, \ldots$ if for every bounded subset $\dot{B}$ in $\hat{\otimes}^n_{s,\pi}E$ there is a bounded subset B in E such that $\dot{B} \subset \overline{co}(\otimes^n_s B)$ ($\overline{co}$ denotes closed convex hull). In section 3 we mention an example of a Fréchet space E such that $\tau_b < \beta$ on $\mathcal{P}(^nE)$ for $n = 2, 3, \ldots$

The last topology we are going to consider on $\mathcal{P}(^nE)$ is the Nachbin ported topology defined as

$$\left(\mathcal{P}(^nE),\ \tau_\omega\right) = \mathrm{ind}_V\, \mathcal{P}(^nE_V)$$

when V ranges over a basis of 0 neighbourhoods in E.

It is easy to see that $\tau_0 \leq \tau_b \leq \beta \leq \tau_\omega$ on $\mathcal{P}(^nE)$. Note that there are spaces E for which some of those topologies are different on $\mathcal{P}(^nE)$ [22]. In section 3 an example of a Fréchet space E such that $\tau_0 < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$ for all $n \geq 2$ will be mentioned. For infinite dimensional Banach spaces we always have $\tau_0 < \tau_b = \beta = \tau_\omega$ on $\mathcal{P}(^nE)$ for all n, so in what follows we only consider locally convex non-Banach spaces.

Those topologies have natural interpretations as linear topologies on $(\hat{\otimes}^n_{s,\pi}E)'$: τ_0 «is» the topology on $(\hat{\otimes}^n_{s,\pi}E)'$ of uniform convergence on the compact subsets of $\hat{\otimes}^n_{s,\pi}E$ which are symmetric n-tensor products of compact subsets of E. In many cases, these compact sets are, modulo absolutely convex closed hulls, the compact, subsets of $\hat{\otimes}^n_{s,\pi}E$. This happens, for instance, when E is metrizable (it follows easily from [33, 41.4] using the symmetrization map, see [26] or [28]). As we have already seen, τ_b «is» the topology on uniform convergence on the bounded subsets of $\hat{\otimes}^n_{s,\pi}E$ of type $\otimes^n_s B$ where B is a bounded subset of E (see [33, 41.5]). β is already defined as the strong topology on $(\hat{\otimes}^n_{s,\pi}E)'$ and τ_ω on $(\hat{\otimes}^n_{s,\pi}E)'$ is the inductive limit $\mathrm{ind}_W(\hat{\otimes}^n_{s,\pi}E)'_{W^0}$ where W ranges over the family of all absolutely convex open subsets of $\hat{\otimes}^n_{s,\pi}E$ (see [26] for the details).

2. THE $\tau_0 = \tau_\omega$ PROBLEM ON $\mathcal{H}(U)$

The first application we mention allowed Taskinen and the first author to solve in the negative the $\tau_0 = \tau_\omega$ problem on $\mathcal{H}(U)$ for Fréchet-Montel spaces. τ_0 is the compact open topology on $\mathcal{H}(U)$ defined as on spaces of polynomials but now the compact subsets are contained in U and τ_ω is defined by the seminorms p on $\mathcal{H}(U)$ such that there exists a compact subset K in U such that for every open neighbourhood V of K in U there is a positive constant C verifying that $p(f) \leq C \sup_{z \in V} |f(z)|$ for all $f \in \mathcal{H}(U)$. This topology on $\mathcal{H}(U)$ induces the τ_ω topology on $\mathcal{P}(^nE)$ [22].

Taskinen constructed in [39] a Fréchet-Montel space F to solve (in the negative) the «Problème des Topologies» (see [31]) and it happens that $F\hat{\otimes}_{s,\pi}F$ contains ℓ_1 as a complemented subspace. Hence $F\hat{\otimes}_{s,\pi}F$ is not a Montel space and then F has not the $(BB)_{2,s}$ property (it is easy to see that if E is a Fréchet-Montel space, then E has the $(BB)_{2,s}$ property if and only if $E\hat{\otimes}_{s,\pi}E$ is a Montel space). As we have already mentioned this implies (in fact it is equivalent) to $\tau_b \neq \beta$ on $\mathcal{P}(^2F)$ and hence $\tau_0 \neq \tau_\omega$ on $\mathcal{P}(^2F)$. Then we have the following

Theorem 1 ([8]). *$\tau_0 \neq \tau_\omega$ on $\mathcal{H}(U)$ for any open subset U of F.*

The above isomorphism has also proved useful in giving partial positive answers to the problem $\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ by using the fact proved by the authors [7] that for balanced open subsets of Fréchet-Montel spaces E, $\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ if and only if $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$ for every $n \in \mathbb{N}$. In this direction results by the authors [7], Dineen [23, 25], Galindo-García-Maestre [29] and Defant-Maestre [19] should be mentioned. All these results concern scalar-valued functions. In the vector-valued case an interesting result follows by combining results by Bierstedt-Bonet-Peris [11] and Boyd-Peris [18] and states that there are Fréchet-Schwartz spaces such that $\tau_0 \neq \tau_\omega$ on $\mathcal{H}(U, X)$ for some open subset U of them and some Banach space X. This contrasts with the scalar case where it is known from Mujica [35] that always $\tau_0 = \tau_\omega$ on $\mathcal{H}(U)$ for balanced open subsets of Fréchet-Swchartz spaces.

3. THE QUASINORMABILITY OF $(\mathcal{P}(^nE), \tau)$

Quasinormable spaces have been introduced by Grothendieck ([30]) as a class of spaces which share the Banach space property: for every equicontinuous subset χ in the dual there is a neighbourhood V of the origin such that on χ, the strong topology and the topology of uniform convergence on V agree. Most function spaces are quasinormable.

When is $(\mathcal{P}(^nE), \tau)$ quasinormable? For $\tau = \tau_0$ it is quasinormable for any Fréchet space. This result has been obtained by Nelimarkka [36] using operator ideals and reobtained by Dineen [24] using a different technique; both proofs are not elementary. Let us see how the isomorphism we are considering gives an easy proof: If E is a Fréchet space, then $\hat{\otimes}^n_{s,\pi}E$ is also a Fréchet space, so its dual with the topology of uniform convergence on compact subsets is quasinormable (these kind of spaces are gDF spaces and hence quasinormable [32]), but this space is $(\mathcal{P}(^nE), \tau_0)$. Note that we are using the fact, already mentioned, that when E is a Fréchet space the compact subsets of $\hat{\otimes}^n_{s,\pi}E$ are liftable.

For $\tau = \tau_b$ the situation, as far as we know, is not completely solved yet: The following result from Bonet-Peris

[17, Sc. 2&3] (see [37]. Ex 1.3.3 for the details), has been proved using the isomorphism (*).

Theorem 2 ([17]). *There exists a Fréchet space such that* $(\mathcal{P}(^nE), \tau_b)$ *is not quasinormable.*

On the other hand, several examples of Fréchet spaces E such that $(\mathcal{P}(^nE), \tau_b)$ is quasinormable are known. We should say that apart from the particular example we are going to comment on now, all Fréchet spaces E for which it is known that $(\mathcal{P}(^nE), \tau_b)$ is quasinormable are spaces such that τ_b agrees with some of the other topologies. For instance, for spaces with the $(BB)_{n,s}$ property, $\tau_b = \beta$ (see [2]). This class includes the spaces with unconditional basis of type (T) and the density condition, in this case $\tau_b = \tau_\omega$ (see [25]). Basis of type (T) have been introduced by Taskinen [39]. Note that $(\mathcal{P}(^nE), \beta)$ is quasinormable because

$$(\mathcal{P}(^nE), \beta) \cong (\hat{\otimes}^n_{s,\pi} E)'_\beta$$

and since $\hat{\otimes}^n_{s,\pi} E$ is a Fréchet space, its strong dual is DF and hence quasinormable (see [32]). For $\tau = \tau_\omega$,

$$(\mathcal{P}(^nE), \tau_\omega) = \text{ind}_{V_k} \mathcal{P}(^nE_{V_k})$$

and every countable inductive limit of a sequence of Banach spaces is DF [30], and hence quasinormable. ((V_k) denotes a countable basis of 0 neighbourhoods in E.)

So the quasinormability of $(\mathcal{P}(^nE), \tau)$ is clear for $\tau = \tau_0$, β and τ_ω, and as we have mentioned before there are Fréchet spaces such that $(\mathcal{P}(^nE), \tau_b)$ is not quasinormable. What follows is an example of a Fréchet space such that $(\mathcal{P}(^nE), \tau_b)$ is quasinormable but $\tau_b \neq \tau_0$ and $\tau_b \neq \beta$. In fact this space has the property that $\tau_0 < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$. The example is due to Blasco and the authors [3]. It is modeled on a Taskinen space similar to the one mentioned previously: If we take a Fréchet-Montel space F such that $F \hat{\otimes}_{s,\pi} F$ contains a non-distinguished $\lambda^1(A)$ space (Taskinen proved that such spaces exist [40]), then for $E = \ell_1 \hat{\otimes}_\pi F$ we have the following result.

Theorem 3 ([3]). *For the above E,* $(\mathcal{P}(^nE), \tau_b)$ *is quasinormable and* $\tau_0 < \tau_b < \beta < \tau_\omega$ *on* $\mathcal{P}(^nE)$ *for n = = 2, 3, ...*

We remark that $(\mathcal{P}(^nE), \tau_b)$ is quasinormable when $E = = \ell_1 \hat{\otimes}_\pi F$ with an arbitrary Fréchet-Montel space F (the particular choice of F is in order to have $\tau_0 < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$). To get the quasinormability of $(\mathcal{P}(^nE), \tau_b)$ we use among other things the fact that $\mathcal{L}_b(^nE)$ is isomorphic to $\ell_\infty((\hat{\otimes}^n_\pi F)', \tau_0)$ (note that F is a Fréchet-Montel space). The fact that all the natural topologies we consider on $\mathcal{P}(^nE)$ are different follows from results in [13] and the fact that $F \hat{\otimes}_\pi F$ contains a non-distinguished space (for details see [3]). It is possible to consider another Fréchet space, similar to the above, such that theorem 3 is also true for $n = 1$, with the natural exception $\tau_b = \beta$ on E' [16].

For a Fréchet E the space $\mathcal{H}_b(U)$ of all holomorphic functions of bounded type on a balanced open subset U of E endowed with its natural topology of uniform convergence on the bounded subsets of E which are strongly contained in U, is quasinormable if and only if $\mathcal{P}(^nE)$ is quasinormable for all n [24, Prop 3.6]. So for the space E in Proposition 5, $\mathcal{H}_b(U)$ is quasinormable for every balanced open subset U of E.

4. BARRELLEDNESS AND REFLEXIVITY OF $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$

For a Köthe matrix A and $1 \leq p < \infty$ let us denote by $\lambda_p(A)$ the corresponding Köthe sequence space. In this section we will be interested in knowing under which conditions on A, p and n the space $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ is barrelled or even reflexive.

We recall that a Köthe matrix A satisfies a certain condition (D) on its coefficients, if and only if the Köthe sequence space $\lambda_p(A)$ satisfy the Heinrich density condition [10].

Theorem 4 ([14]). *If the Köthe matrix A does not satisfy the density condition (D) and* $1 < p < \infty$, *then* $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ *is barrelled if and only if* $n < p$.

The key fact is that, for $n < p$, the space $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ is reflexive (independent of A) [14] and hence distinguished. Note that in this case $\tau_b = \beta$ because Köthe spaces have the $(BB)_{n,s}$ property [23]. If $n \geq p$ then a copy of a certain non-distinguished λ_1 is complemented in $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ [14], which implies that this space is not distinguished.

From the above result it follows that $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ is barrelled for all $n \in \mathbb{N}$ if and only if A satisfies condition (D) [14].

We recall that a Köthe matrix A satisfies condition (M) if and only if the Köthe sequence space $\lambda_p(A)$ is a Montel space [12]).

Theorem 5 ([14]). *For* $1 \leq p < \infty$, $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ *is reflexive for all n if and only if A satisfies condition (M).*

In this situation the reflexivity of $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ is equivalent to being Montel. The key is that when A satisfies condition (M), $\lambda_p(A)$ is Montel and hence $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ is Montel (the spaces $\lambda_p(A)$ have the $(BB)_{n,s}$ property). This implies that its strong dual is Montel, hence reflexive (note that $\tau_b = \beta$). On the other hand if A does not satisfy the condition (M) and $n \geq p$, $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ is not reflexive because a certain non-distinguished λ_1 is complemented in $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ [14]. This λ_1 can be chosen so that its matrix does not satisfy the condition (M) and hence it contains a complemented subspace isomorphic to ℓ_1 (see [41]). Hence $\hat{\otimes}^n_{s,\pi} \lambda_p(A)$ cannot be reflexive. Note finally that this implies, by duality, that $(\mathcal{P}(^n\lambda_p(A)), \tau_b)$ is not reflexive.

Remark. *The above result has also been obtained by Dineen-Lindström in* [27] *in a different way.*

Given a Banach space X with a basis (e_n) *and a Köthe matrix* $A = (a_{m,n})$ *consider the following space defined by Bellenot* [9]:

$$\lambda_X(A) = \{(x_n) \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_{n=1}^{\infty} a_{m,n}x_ne_n \in X \text{ for each } m \in \mathbb{N}\}$$

($\lambda_X(A)$ is a Fréchet space called a X-Köthe sequence space.)

Theorem 6 ([15]). *If X is a Banach space with an 1-unconditional basis, then* $\hat{\otimes}^n_{s,\pi}\lambda_X(A)$ *is reflexive for every matrix A provided that* $\hat{\otimes}^n_{s,\pi}X$ *is reflexive, and then* $(\mathcal{P}(^n\lambda_X(A)), \tau_b)$ *is reflexive when* $(\mathcal{P}(^nX), \tau_b)$ *is.*

This can be applied for instance to show that $(\mathcal{P}(^n\lambda_{T'}(A)), \tau_b)$ is reflexive (T' is the original Tsirelson space). Note that in [1] it is proved that $(\mathcal{P}(^nT), \tau_b)$ is reflexive for all n. If we take a matrix A without condition (M) still $(\mathcal{P}(^n\lambda_{T'}(A)), \tau_b)$ is reflexive for all n (compare this result with the analogous for $X = \ell_p$ in Theorem 5). For the details see [15].

5. POLYNOMIALS ON STABLE SPACES

Let E be a stable locally convex space (i.e. $E \cong E \times E$), then $(\mathcal{P}(^nE), \tau_b)$ and $\mathcal{L}_b(^nE)$ are isomorphic. This result has been obtained by Díaz-Dineen in [21] and can also been obtained from the following

Theorem 7 ([6]). *If* F_1 *and* F_2 *are two locally convex spaces, then*

$$\hat{\otimes}^n_{s,\tau}(F_1 \oplus F_2) \cong \oplus^n_{k=0}[\hat{\otimes}^k_{s,\tau}F_1]\hat{\otimes}_\tau[\hat{\otimes}^{n-k}_{s,\tau}F_2].$$

for all symmetric tensor topologies (including ε and π).

For instance, for $n = 3$ the isomorphism is given by:

$$\otimes^3(x_1, x_2) \to$$

$$\to (x_2 \otimes x_2 \otimes x_2, x_1 \otimes x_2 \otimes x_2, x_1 \otimes x_1 \otimes x_2, x_1 \otimes x_1 \otimes x_1).$$

When E is a stable locally convex space we can write $E = F_1 \oplus F_2$ with $F_1 = F_2 = E$ and then the formula gives

$$\hat{\otimes}^n_{s,\tau}E \cong \oplus^n_{k=0}[\hat{\otimes}^k_{s,\tau}E]\hat{\otimes}_\tau[\hat{\otimes}^{n-k}_{s,\tau}E]$$

and from it, one obtains that $\hat{\otimes}^n_{s,\tau}E$ and $\hat{\otimes}^n_\tau E$ are topologically isomorphic, this implies (by considering $\tau = \pi$) and using the isomorphism (*) that $(\mathcal{P}(^nE), \tau_b) \cong \mathcal{L}_b(^nE)$ which is the Díaz-Dineen result mentioned before.

6. BARRELLEDNESS OF $(\mathcal{P}(^nE; F), \tau_\omega)$

The space $(\mathcal{P}(^nE, F), \tau_\omega)$ is always ultrabornological when E is a Fréchet space and F is a Banach space. It is a countable inductive limit of Banach spaces. This is not in general the case when we consider polynomials with values in a Fréchet space. Mangino in her thesis, using the isomorphism (*), found necessary and sufficient conditions for the space $(\mathcal{P}(^nE; F), \tau_\omega)$ to be ultrabornological. When we consider polynomials, with values in a Fréchet space, τ_ω is the projective limit of the spaces $(\mathcal{P}(^nE; F_{W_n}), \tau_\omega)$ where F_{W_n} are the normed spaces asociated to a countable neighborhood basis (W_n) of 0 in F.

Theorem 8 ([34]). *Let E and F be Fréchet spaces, one of which is nuclear, then* $(\mathcal{P}(^nE; F), \tau_\omega)$ *is ultrabornological if and only if the functor* $Ext^1(\hat{\otimes}^n_{s,\pi}E, F)$ *vanishes.*

This means that for every exact short sequence $0 \to F \to G \to \hat{\otimes}^n_{s,\pi}E \to 0$, q has an inverse (q is the map between G and $\hat{\otimes}^n_{s,\pi}E$). Other hypotheses, different to nuclearity, can be placed on the spaces and the theorem remains true. In general, if $Ext^1(\hat{\otimes}^n_{s,\pi}E, F) = 0$ then $(\mathcal{P}(^nE; F), \tau_\omega)$ is ultrabornological (see [34] for details).

7. THE THREE-SPACE PROBLEM FOR SPACES OF POLYNOMIALS

The isomorphism (*) also has applications to the problem of whether equality between topologies on $\mathcal{P}(^nE)$ is a three-space property. That is, suppose E is a locally convex space, F is a closed subspace of E and two of the topologies τ_0, τ_b, β or τ_ω agree on $\mathcal{P}(^nF)$ and on $\mathcal{P}(^nE/F)$. Do they agree on $\mathcal{P}(^nE)$?

The equality between τ_0 and τ_b is a three-space property [4], but this is not the case for the equality between τ_b and β. In proving this the isomorphism (*) is used:

Theorem 9 ([4]). *If* E_1 *is the projective limit* $l_{p+} = \cap_{q>p}l_q$, *with* $2 \le p < \infty$ *and* $E_2 = C_2$, *the* l^2*-sum of a sequence of finite dimensional Banach spaces dense in the set of all finite dimensional spaces with respect to the Banach-Mazur distance, then* $\tau_b = \beta$ *on* $\mathcal{P}(^nE_1)$ *and on* $\mathcal{P}(^nE_2)$ *but* $\tau_b \neq \beta$ *on* $\mathcal{P}(^n(E_1 \times E_2))$.

So the equality $\tau_b = \beta$ is not a three-space property even for complemented subspaces. Using again the isomorphism (*) the following result can be obtained:

Theorem 10 ([4]). *Let* E_1 *be the Banach space* l_1 *and* E_2 *the Köthe sequence space* $\lambda_p(A)$ *where A is the Grothendieck-Köthe matrix and p is a real number bigger than n (a given natural number). Then* $\beta = \tau_\omega$ on $\mathcal{P}(^nE_1)$ *and on* $\mathcal{P}(^nE_2)$ *but* $\beta \neq \tau_\omega$ *on* $\mathcal{P}(^n(E_1 \times E_2))$.

So the equality $\beta = \tau_\omega$ is not a three-space property even for complemented subspaces. Using any of the above theorems it can be shown that the equality $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$ is not a three-space property even for complemented subspaces.

8. COINCIDENCE OF TOPOLOGIES ON $\mathcal{P}(^nE)$ FOR HIGHER DEGREES

Another problem in which the isomorphism (*) has succeeded is the problem of determining if two of the topologies on $\mathcal{P}(^nE)$ agree for all n if coincide for a finite number of n. Of course, when they coincide for a certain n they agree for every m, $m \leq n$ [13].

This problem, for the τ_b and β topologies is again very much related with property $(BB)_{n,s}$. In fact, if we have an space E with a certain property $(BB)_{n,s}$ but without property $(BB)_{m,s}$, for some $m > n$, then $\tau_b = \beta$ on $\mathcal{P}(^nE)$ but $\tau_b \neq \beta$ on $\mathcal{P}(^mE)$.

The counterexample is built on the space l_{p+}, considered in section 6, and a Banach space X such that the couple (l_{p+}, X') does not have the (BB) property. This space is constructed in [20]. For the space $E = l_{p+} \times X'$ we have the following:

Theorem 11 [5]. *For $1 < p < 2$ and X as above the space $E = l_{p+} \times X'$ has property $(BB)_2$ but does not have property $(BB)_{3,s}$.*

Corollary 12 [5]. *For the space E in Theorem 11 we have $\tau_b = \beta$ on $\mathcal{P}(^2E)$ but $\tau_b \neq \beta$ on $\mathcal{P}(^3E)$ and also $\tau_b = \tau_\omega$ on $\mathcal{P}(^2E)$ but $\tau_b \neq \tau_\omega$ on $\mathcal{P}(^3E)$.*

REFERENCES

1. Alencar, R., Aron, R. & Dineen, S. (1984). A reflexive space of holomorphic functions in infinitely many variables, *Proc. Amer. Math. Soc.* **90**, 407-411.
2. Ansemil, J. M. (1994). On the quasi-normability of $H_b(U)$, *Extracta Math.* **9**, 1, 71-74.
3. Ansemil, J. M., Blasco, F. & Ponte, S. (1997). Quasi-normability and topologies on spaces of polynomials, *J. Math. Anal. Appl.* **213**, 534-539.
4. Ansemil, J. M., Blasco, F. & Ponte, S. (1997). On the «Three-Space Problem» for spaces of polynomials, Proc. II international Workshop on Functional Analysis at Trier University. Note Mat. **17**, 189-195.
5. Ansemil, J. M., Blasco, F. & Ponte, S. (2000) *(BB)* properties on Fréchet spaces, *Ann. Acad. Sci. Fenn. Math.* **25,** 307-316.
6. Ansemil, J. M. & Floret, K. (1998). The symmetric tensor product of a direct sum of locally convex spaces, *Studia Math.* **129**, 3, 285-295.
7. Ansemil, J. M. & Ponte, S. (1988). The compact open topology and the Nachbin ported topology on spaces of holomorphic functions, *Arch. Math. (Basel)* **51**, 65-70.
8. Ansemil, J. M. & Taskinen, J. (1990). On a problem of topologies in infinite dimensional holomorphy, *Arch. Math. (Basel)* **54**, 61-64.
9. Bellenot, S. (1980). Basic sequences in non-Schwartz spaces, *Trans. Amer. Math. Soc.* **258**, 199-216.
10. Bierstedt, K. D. & Bonet, J. (1989). Density condition in Fréchet and (DF)-spaces, *Rev. Mat. Univ. Compl. Madrid* **2**, 59-75.
11. Bierstedt, K. D., Bonet, J. & Peris, A. (1994). Vector-valued holomorphic germs on Fréchet-Schwartz spaces, *Proc. Roy. Irish Acad. Sect. A* **94**, 1, 31-46.
12. Bierstedt, K. D., Meise, R. & Summers, B. (1982). Köthe sets and Köthe sequence spaces, *North-Holland Math. Stud.* **71**, 27-91.
13. Blasco, F. (1997). Complementation in spaces of symmetric tensor products and polynomials, *Studia Math.* **123**, 2, 165-173.
14. Blasco, F. (1998). Polynomials on Köthe echelon spaces, *Arch. Math. (Basel)* **70**, 147-152.
15. Blasco, F. (1998). On *X*-Köthe echelon spaces and applications, *Proc. Roy. Irish Acad. Sect. A,* **2**, 191-208.
16. Blasco, F. (2000). Quasinormability of vector valued sequence spaces, Proc. of Finite and Infinite Dimensional Complex Analysis, 7th. Int. Colloquium, Fukuoka. Lec. Notes in Pure and Applied Series. Marcel Dekker, 21-28.
17. Bonet, J. & Peris, A. (1991). On the injective tensor product of quasinormable spaces, *Results in Math.* **20**, 431-443.
18. Boyd, C. & Peris, A. (1996). A projective description of the Nachbin-ported topology, *J. Math. Anal. Appl.* **197**, 3, 635-657.
19. Defant, A. & Maestre, M. (1994). Property *(BB)* and holomorphic functions on Fréchet-Montel spaces, *Math. Proc. Cambridge Philos. Soc.,* **115**, 2, 305-313.
20. Defant, A. & Peris, A. (1998). Maurey's extension theorem and Grothendieck's «problème des topologies», *J. London Math. Soc.* **58**, 2, 679-696.
21. Díaz, J. C. & Dineen, S. (1998). Polynomials on stable Banach spaces, *Ark. Mat.* **36**, 1, 87-96.
22. Dineen, S. (1981). Complex Analysis on Locally Convex Spaces, *North-Holland Math. Stud.* **57**.
23. Dineen, S. (1992). Holomorphic functions on Fréchet Montel spaces, *J. Math. Anal. Appl.,* **163**, 581-587.
24. Dineen, S. (1993). Quasi-normable spaces of holomorphic functions, *Note Mat.,* **12**, 1, 155-195.
25. Dineen, S. (1994). Holomorphic functions and the *(BB)*-property, *Math. Scand.* **74**, 215-236.
26. Dineen, S. (1999). Complex Analysis on Infinite Dimensional Spaces, *Springer Monographs in Mathematics,* Springer.
27. Dineen, S. & Lindström, M. (1996). *Spaces of homogeneous polynomials containing c_0 or l^∞, Functional Analysis,* Walter de Gruyter & Co, Berlin, 183-194.
28. Floret, K. (1997). Natural norms on symmetric tensor products of normed spaces, Proc. II international Workshop on Functional Analysis at Trier University. Note Mat. **17,** 153-188.
29. Galindo, P., García, D. & Maestre, M. (1991). The coincidence of τ_0 and τ_ω for spaces of holomorphic functions on some Fréchet-Montel spaces, *Proc. Roy. Irish Acad. Sect. A* **91**, 2, 137-143.
30. Grothendieck, A. (1954). Sur les espaces (F) and (DF), *Summa Brasihensis Math.* **3**, 6, 57-123.
31. Grothendieck, A. (1955). Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.* **16**.

32. Jarchow, H. (1981). *Locally Convex Spaces,* B. G. Teubner, Stuttgart.

33. Köthe, G. (1979). «Topological vector spaces», II, Springer-Verlag.

34. Mangino, E. (1997). Productos tensoriales de espacios (LF), (DF) y de Fréchet, Thesis. Univ. of Valencia.

35. Mujica, J. (1984). A Banach-Dieudonné theorem for germs of holomorphic functions, *J. Funct. Anal.* **57**, 1, 31-48.

36. Nelimarkka, E. (1980). On spaces of holomorphic functions on locally convex spaces defined by an operator ideal. Notes on Funct. Anal. II, Ed. L. Holmstrom, Univ. of Helsinki, 25-35.

37. Peris, A. (1992). Productos tensoriales de espacios localmente convexos y otras clases relacionadas. Thesis, Univ. of Valencia.

38. Ryan, R. (1980). Applications of topological tensor products to infinite dimensional holomorphy, Thesis, Trinity College Dublin.

39. Taskinen, J. (1986). Counterexamples to «Problème des topologies» of Grothendieck, *Ann. Acad. Sci. Fenn. Math. Diss.* **63**.

40. Taskinen, J. (1989). Examples of non-distinguished Fréchet spaces, *Ann. Acad. Sci. Fenn.* **14**, 75-88.

41. Valdivia, M. (1982). Topics in Locally Convex Spaces, *North-Holland Math. Studies* **67**.