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ON *K*-NEARLY UNIFORM CONVEXITY IN ORLICZ SPACES[1]

(Orlicz spaces, *kNUC* property, Banach-Saks Property, Weak Banach-Saks Property)

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ABSTRACT

It is shown that each Banach space with property (*kNUC*) has the Banach-Saks property. As a consequence of this result it is noticed that there exists a Banach space which is (*NUC*) but not (*kNUC*). Criteria for property (*kNUC*) in Orlicz function spaces and Orlicz sequence spaces are given. In Orlicz function spaces property (*kNUC*) coincide with uniform convexity. In a contrast to this result, in Orlicz sequence spaces property (*kNUC*) is essentially weaker than uniform convexity and it is equivalent to reflexivity.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a real Banach space, and let X^* be the dual space of X. Let $B(X)$ $\big(S(X)\big)$ be a closed unit ball (a unit sphere) of X.

In 1937, J. A. Clarkson [4] introduced the concept of uniform convexity.

A Banach space X is called uniformly convex (write (**UC**)) (see [4], [5], [6], [14] and [20]) if for each $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ the inequality $\|x - y\| > \varepsilon$ implies

$$\left\|\frac{1}{2}(x + y)\right\| < 1 - \delta.$$

Let $k \geq 2$ be an integer. Recall that a Banach space X is said to be fully k-rotund $\big((kR)$ for short$\big)$ if for every sequence $\{x_n\} \subset B(X)$, $\|x_{n_1} + \cdots + x_{n_k}\| \to k$ as $n_1, n_2, \ldots, n_k \to \infty$ implies that $\{x_n\}$ is a Cauchy sequence (see [7]).

It is well known that $(UR) \Rightarrow (kR) \Rightarrow \big((k+1)R\big)$ and (kR) space is reflexive and rotund.

The next notion is a generalization of the nearly uniform convexity $\big((NUC)$ for short$\big)$ introduced by Huff (see [9]). For an integer $k \geq 2$, a Banach space X is said to be *compactly fully k-rotund* (CkR) if for every sequence $\{x_n\} \subset B(X)$, $\|x_{n_1} + x_{n_2} + \cdots + x_{n_k}\| \to k$ as $n_1, n_2, \ldots, n_k \to \infty$ implies that $\{x_n\}$ is a relatively compact sequence (see [13]).

If $k \geq 2$ is an integer, a Banach space X is said to be ($kNUC$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every sequence $\{x_n\} \subset B(X)$ with $\mathrm{sep}(x_n) := = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$ there are $n_1\, n_2, \ldots, n_k \in N$ for which $\left\|\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}\right\| < 1 - \delta$ (see [12]).

A Banach space X is said to have the weak Banach-Saks property if every weakly null sequence $\{x_n\}$ in the unit ball $B(X)$ of X admits a subsequence $\{z_n\}$ such that the sequence of the arithmetic means $\left\{\frac{1}{n}(z_1 + z_2 + \cdots + z_n)\right\}$ is convergent in X.

Denote by $\mathcal{N}$ and $\mathcal{R}$ the sets of natural and real numbers, respectively. Let (G, Σ, μ) be a measure space with a finite measure μ. Denote by L^0 the set of all μ-equivalence classes of real valued measurable functions defined on G. Let l^0 stand for the space of all real sequences.

A map $\Phi\colon \mathcal{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is vanishing at 0, even, convex and not identically equal to 0. An Orlicz function is called an *N-function* at ∞ (resp. at 0) if

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$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty \left(\text{resp. } \lim_{u\to 0} \frac{\Phi(u)}{u} = 0\right).$$

By the *Orlicz function space* L_Φ we mean

$$L_\Phi = \left\{x \in L^0 : I_\Phi(cx) = \int_G \Phi\big(cx(t)\big)\, d\mu, < \infty \textit{ for some } c > 0\right\}.$$

Analogously, we define the *Orlicz sequence space* by the formula

$$l_\Phi = \left\{x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi\big(cx(i)\big) < \infty \textit{ for some } c > 0\right\}.$$

L_Φ and l_Φ are equipped with the so called *Luxemburg norm*

$$\|x\| = \inf\left\{\varepsilon > 0 : I_\Phi\left(\frac{x}{\varepsilon}\right) \le 1\right\}$$

or with equivalent one

$$\|x\|_0 = \inf_{k>0} \frac{1}{k}\big(1 + I_\Phi(kx)\big)$$

called the *Orlicz* or the *Amemiya norm.* It is well known that if Φ is an N-function, then for every $x \neq 0$ there exists $k > 0$ such that

$$\|x\|_0 = \frac{1}{k}\big(1 + I_\Phi(kx)\big).$$

To simplify notations, we put $L_\Phi = (L_\Phi, \|\cdot\|)$, $l_\Phi = (l_\Phi, \|\cdot\|_0)$, $L_\Phi^0 = (L_\Phi, \|\cdot\|_0)$ and $l_\Phi^0 = (l_\Phi^0, \|\cdot\|_0)$.

For every Orlicz function Φ we define its *complementary function* $\Psi : \mathcal{R} \to [0, \infty)$ by the formula

$$\Psi(v) = \sup_{u>0} \{u|v| - \Phi(u)\}$$

for every $v \in \mathcal{R}$.

We say an Orlicz function Φ *satisfies the* Δ_2*-condition* {δ_2*-condition*) if there exist constants $k \ge 2$ and $u_0 > 0$ such that $\Phi(u_0) > 0$ and

$$\Phi(2u) \le k\Phi(u)$$

for every $|u| \ge u_0$ (for every $|u| \le uy$), respectively.

We say an Orlicz function Φ *satisfies the* ∇_2-condition ($\bar{\delta}_2$-condition) if its complementary function Ψ satisfies the Δ_2-condition (δ_2-condition), respectively.

An Orlicz function Φ is said to be *uniformly convex* on $[0, u_0]$, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Phi\left(\frac{u+v}{2}\right) \le (1 - \delta)\, \frac{M(u) + M(v)}{2}$$

for all $u,\ v \in [0, u_0]$ satisfying $|u - v| \ge \epsilon \max\{u, v\}$.

We say an Orlicz function Φ is *strictly convex* if for any $u \neq v$ and $\alpha \in (0, 1)$ we have

$$\Phi(\alpha u + (1 - \alpha)v) < \alpha\Phi(u) + (1 - \alpha)\Phi(v).$$

For the above informations and more details on Orlicz functions and Orlicz spaces we refer to [2], [11], [16] or [17].

2. RESULTS

Theorem 1. *Let X be a Banach space. If there exists* $\theta \in (0, 1)$ *such that for every weakly null sequence* $\{x_n\} \subset B(X)$, *there exist* $n_1, n_2, \ldots, n_k \in N$ *for some* $k \in N$ *for which*

$$\left\|\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{h}\right\| < 1 - \theta,$$

then X has the weak Banach-Saks property.

Proof. For every weakly null sequence $\{x_n\} \subset B(X)$, there exist $n_1^{(1)}, n_2^{(1)}, \ldots, n_k^{(1)} \in N$ such that

$$\left\|\frac{x_{n_1}^{(1)} + x_{n_2}^{(1)} + \cdots + x_{n_k}^{(1)}}{k}\right\| < 1 - \theta.$$

For the weakly null sequence $\{x_n\}_{n > n_k^{(1)}}$, there exist also $n_1^{(2)}, n_2^{(2)}, \ldots, n_k^{(2)} \in N$ such that

$$\left\|\frac{x_{n_1}^{(2)} + x_{n_2}^{(2)} + \cdots + x_{n_k}^{(2)}}{k}\right\| < 1 - \theta.$$

In such a way, we can get a system $\{x_{n_1}^{(i)}, x_{n_2}^{(i)}, \ldots, x_{n_k}^{(i)}\}_{i=1}^{\infty}$ of subsequences of $\{x_n\}$ such that

$$\left\|\frac{x_{n_1}^{(i)} + x_{n_2}^{(i)} + \cdots + x_{n_k}^{(i)}}{k}\right\| < 1 - \theta$$

for each $i \in N$.

Put $y_i = \dfrac{x_{n_1}^{(i)} + x_{n_2}^{(i)} + \cdots + x_{n_k}^{(i)}}{k(1-\theta)}$. Then $\{y_i\}$ is a weakly null sequence in $B(X)$. In the same way as above, we can get a system $\{y_{m_1}^{(i)}, y_{m_2}^{(i)}, \ldots, y_{m_k}^{(i)}\}_{i=1}^{\infty}$ of subsequences of $\{y_n\}$ such that

$$\left\|\frac{y_{m_1}^{(i)} + y_{m_2}^{(i)} + \cdots + y_{m_k}^{(i)}}{k}\right\| < 1 - \theta$$

for each $i \in N$. Hence, for each $m_1, \ldots, m_k \in \mathbb{N}$

$$\frac{1}{k}\left\|\frac{x_{n_1}^{(m_1)} + \cdots + x_{n_k}^{(m_1)}}{k(1-\theta)} + \cdots + \frac{x_{n_1}^{(m_k)} + \cdots + x_{n_k}^{(m_k)}}{k(1-\theta)}\right\| < 1 - \theta,$$

i.e.,

$$\frac{1}{k^2}\|x_{n_1}^{(m_1)} + \cdots + x_{n_k}^{(m_1)} + \cdots + x_{n_1}^{(m_k)} + \cdots + x_{n_k}^{(m_k)}\| < (1-\theta)^2.$$

Let $\varepsilon > 0$ be given. There exists $l \in \mathcal{N}$ such that $(1-\theta)^l < \varepsilon$. Repeating the above procedure l times, we have

$$\frac{1}{k^l}\left\|\sum_{j=1}^{k^{l-1}}\left(x_{n_1}^{(i_j)} + x_{n_2}^{(i_j)} + \cdots + x_{n_k}^{(i_j)}\right)\right\| < (1-\theta)^l < \varepsilon.$$

By Theorem 2 in [18], we get that X has the weak Banach-Saks property. □

Corollary. *If a Banach space X is (kNUC) for some $k \geq 2$, then it has the Banach-Saks property.*

Proof. Notice that X with property *(kNUC)* is reflexive and that X has the Banach-Saks property if and only if X is reflexive and it has the weak Banach-Saks property. So, by Theorem 1, the corollary follows. □

Remark. *There is a Banach space X which is (NUC) but it is not (kNUC) for any $k \geq 2$, $k \in \mathcal{N}$.*

Proof. Kutzarova [12] has shown that if for some $k \geq 2$, $k \in \mathcal{N}$, X is *(kNUC)* then X is *(NUC)*. It is well known that the *Baernstein* space B is *(NUC)* (see [1] and [20]) and it has not the Banach-Saks property (see [20]). So by Corollary, we know that the *Baernstein* space is not *(kNUC)* for any $k \geq 2$, $k \in \mathcal{N}$. This means that property *(kNUC)* is essentially stronger than property *(NUC)*. □

Theorem 2. *Let Φ be a Orlicz function. L_Φ (or L_Φ^0) is (kNUC) if and only if Φ is a strictly convex Orlicz function satisfying the Δ_2-condition and Φ is uniformly convex outside a neighbourhood of zero.*

Proof. The sufficiency follows from the fact that L_Φ (or L_Φ^0) is *(UC)* under the assumption that Φ is a strictly convex Orlicz function satisfying the Δ_2-condition and Φ is uniformly convex outside a neighbourhood of zero. We only need to prove the necessity of the theorem.

Obviously, every Banach space X with property *(kNUC)* has the *Kadec-Klee* property.

In the case of L_Φ, the proof of the necessity of strict convexity of Φ for the *Kadec-Klee* property we can find in [19]. Next, we will show that Φ must be strictly convex in the case when L_Φ^0 has the *Kadec-Klee* property.

Assume the contrary, i.e. there is an interval $[a, b]$ such that right-hand derivative p of Φ is constant on $[a, b]$. Take $G^0 \subset G$ such that $0 < \mu(G \setminus G^0) < \mu(G)$ and choose $c > 0$ and $G' \subset G \setminus G^0$ satisfying

$$\Psi(p(a))\mu(G^0) + \Psi(p(c))\mu(G') = 1.$$

Divide G^0 into two subsets G_1^1 and G_2^1 such that $G^0 = G_1^1 \cup G_2^1$ with $\mu(G_1^1) = \mu(G_2^1)$. Suppose that the sequence of sets $(G_1^{n-1}, G_2^{n-1}, \ldots, G_{2^{n-1}}^{n-1})$ is already defined. Every set G_i^{n-1} we divide into two subsets G_{2i-1}^n, G_{2i}^n such that $G_i^{n-1} = G_{2i-1}^n \cup G_{2i}^n$ and $\mu(G_{2i-1}^n) = \mu(G_{2i}^n)$ $(i = 1, 2, \ldots, 2^{n-1})$. In such a way, we obtain a system of partitions $(G_1^n, G_2^n, \ldots, G_{2^n}^n)$ of G^0 such that

$$\mu(G_i^n) = 2^{-n}\mu(G^0) \quad (i = 1, 2, \ldots, 2^n).$$

Denote

$$k = 1 + \Phi\left(\frac{a+b}{2}\right)\mu(G^0) + \Phi(c)\mu(G')$$

and put

$$x_n = \frac{1}{k}\,(a\chi_{E_{1,n}} + b\chi_{E_{2,n}}),$$

where $E_{1,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k-1}^n$, $E_{2,n} = \bigcup_{k=1}^{2^{n-1}} G_{2k}^n$ $(n = 1, 2, \ldots)$. Since $I_\Psi(p(kx_n)) = 1$, we have

$$\|x_n\|_0 = \frac{1}{k}\left(1 + I_\Phi(kx_n)\right) = 1 \quad (n = 1, 2, \ldots).$$

By the reflexivity of L_Φ^0, we can assume that there exists $x \in B(L_\Phi^0)$ such that $x_n \xrightarrow{w} x$.

Since $v = p\left(\frac{a+b}{2}\chi_{G^0} + c\chi_{G'}\right)$ defines a functional which supports x_n $(n = 1, 2, \ldots)$, so $x \in S(L_\Phi^0)$. But

$$\|x_n - x_m\|_0 = \frac{b-a}{k}\cdot\frac{\mu(G^0)}{2}\,\Psi^{-1}\left(\frac{2}{\mu(G^0)}\right) \quad (n, m = 1, 2, \ldots, n \neq m).$$

This contradicts the Kadec-Klee property.

The necessity of the uniform convexity of Φ outside a neighbourhood of zero and of the Δ_2-condition is proved in [2], Theorem 3.15. □

Theorem 3. *The Orlicz sequence space l_Φ is (kNUC) if and only if Φ satisfies both the δ_2-condition and the $\overline{\delta}_2$-condition, i.e. l_Φ is reflexive.*

Proof. We need only to prove the sufficiency of theorem. Suppose that the implication is not true. Let an arbitrary $\varepsilon > 0$ and any $(x_n) \subset B(l_\Phi)$ with $\text{sep}(x_n) > \varepsilon$ be given. By $\Phi \in \delta_2$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\inf\left\{I_\Phi\left(\frac{x_n - x_m}{2}\right) : n \neq m\right\} \geq \delta.$$

Next, we will show that for any $j \in N$ there exists $n_j \in N$ such that

$$(1) \qquad \sum_{i=j}^{\infty} \Phi(x_{n_j}(i)) \geq \frac{\delta}{3}.$$

Otherwise, there exists $j_0 \in N$ such that

$$\sum_{i=j}^{\infty} \Phi(x_{n_j}(i)) < \frac{\delta}{3}$$

for any $j \in N$.

Defining $\bar{x}_n = (x_n(1), x_n(2), \ldots, x_n(j_0), 0, 0, \ldots)$ for $n \in N$, we easily get that there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ such that

$$I_\Phi\left(\frac{\bar{x}_{n_i} - \bar{x}_{n_j}}{2}\right) < \frac{\delta}{3}$$

for any $i \neq j$. Hence

$$I_\Phi\left(\frac{x_{n_i} - x_{n_j}}{2}\right) = I_\Phi\left(\frac{\sum_{k=1}^{j_0} (x_{n_i}(k) - x_{n_j}(k))\, e_k}{2}\right) +$$

$$+ I_\Phi\left(\frac{\sum_{k=j_0+1}^{\infty} (x_{n_i}(k) - x_{n_j}(k))\, e_k}{2}\right) \leq I_\Phi\left(\frac{\sum_{k=1}^{j_0} (x_{n_i}(k) - x_{n_j}(k))\, e_k}{2}\right) +$$

$$+ \frac{1}{2}\sum_{k=j_0+1}^{\infty} \Phi(x_{n_i}(k)) + \frac{1}{2}\sum_{k=j_0+1}^{\infty} \Phi(x_{n_j}(k)) = I_\Phi\left(\frac{\bar{x}_{n_i} - \bar{x}_{n_j}}{2}\right) +$$

$$+ \frac{1}{2}\sum_{k=j_0+1}^{\infty} \Phi(x_{n_i}(k)) + \frac{1}{2}\sum_{k=j_0+1}^{\infty} \Phi(x_{n_j}(k)) < \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} = \frac{2}{3}\delta < \delta.$$

This contradiction shows that (1) holds.

Since Φ satisfies the $\bar{\delta}_2$-condition, there is $0 < \Theta < 1$ such that

$$(2) \qquad \Phi\left(\frac{u}{k}\right) \leq (1 - \Theta)\,\frac{\Phi(u)}{k} \quad (\forall\, 0 \leq u \leq \Phi^{-1}(1))$$

(see [2], [3] and [8]).

By $\Phi \in \delta_2$, there exists $\theta > 0$ such

$$(3) \qquad |I_\Phi(x + y) - I_\Phi(x)| < \frac{\Theta\delta}{6k}$$

whenever $I_\Phi(x) \leq 1$, $I_\Phi(y) \leq \theta$ (see [2], [10]).

Take $n_1 < n_2 < \cdots < n_{k-1}$, $n_1, n_2, \ldots, n_{k-1} \in N$. Notice that

$$I_\Phi\left(\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_{k-1}}}{k}\right) \leq 1$$

and $I_\Phi(x_{n_i}) \leq 1$ for $i = 1, 2, \ldots, k - 1$. There exists $j_0 \in N$ such that

$$(4) \qquad \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_1}(i) + x_{n_2}(i) + \cdots + x_{n_{k-1}}(i)}{k}\right) < \theta$$

and

$$\sum_{i=j_0+1}^{\infty} \Phi(x_{n_j}(i)) < \frac{\delta}{3} \qquad (j = 1, 2, \ldots, k - 1).$$

By (1), there exists $n_k \in N$ such that

$$(5) \qquad \sum_{i=j_0+1}^{\infty} \Phi(x_{n_k}(i)) \geq \frac{\delta}{3}.$$

So, in virtue of (2), (3), (4) and (5), we get

$$I_\Phi\left(\frac{x_{n_1} + \cdots + x_{n_k}}{k}\right) =$$

$$= \sum_{i=1}^{j_0} \Phi\left(\frac{x_{n_1}(i) + \cdots + x_{n_k}(i)}{k}\right) +$$

$$+ \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_1}(i) + \cdots + x_{n_k}(i)}{k}\right) \leq \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} \Phi(x_{n_j}(i)) +$$

$$+ \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_k}(i)}{k}\right) + \frac{\Theta\delta}{6k} \leq \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j_0} \Phi(x_{n_j}(i)) +$$

$$+ \frac{1 - \Theta}{k}\sum_{i=j_0+1}^{\infty} \Phi(x_{n_k}(i)) + \frac{\Theta\delta}{6k} = \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{\infty} \Phi(x_{n_j}(i)) -$$

$$- \frac{\Theta}{k}\sum_{i=j_0+1}^{\infty} \Phi(x_{n_k}(i)) + \frac{\Theta\delta}{6k} \leq 1 - \frac{\Theta\delta}{3k} + \frac{\Theta\delta}{6k} = 1 - \frac{\Theta\delta}{3k}.$$

This completes the proof. □

Theorem 4. *For any N-function* Φ *at* 0 *the Orlicz sequence spaces* l_Φ^0 *is (kNUC) if and only if* Φ *satisfies both the* δ_2*-condition and the* $\bar{\delta}_2$*-condition, i.e.* l_Φ^0 *is reflexive.*

Proof. We only need to prove the sufficiency. Let an $\varepsilon > 0$ and any $(x_n) \subset B(l_\Phi^0)$ with $\mathrm{sep}(x_n) > \varepsilon$ be given. By $\Phi \in \delta_2$, there exists $\delta > 0$ such that

$$\inf\left\{I_\Phi\left(\frac{x_n - x_m}{2}\right) : n \neq m\right\} \geq \delta.$$

By the arguments as the Theorem 3, we have that for any $j \in N$ there exists $n_j \in N$ such that

$$\sum_{i=j}^{\infty} \Phi\big(x_{n_j}(i)\big) \geq \frac{\delta}{3}. \tag{6}$$

Take $k_n \geq 1$ such that

$$\|x_n\|_0 = \frac{1}{k_n}\big(1 + I_\Phi(k_n x_n)\big).$$

Since Φ satisfies the $\bar{\delta}_2$-condition, the number

$$k_0 = \sup\{k_n : n = 1, 2, \ldots\}$$

is finite (see [2]). Fix $n_1 < n_2 < \cdots < n_{k-1}, n_1, n_2, \ldots, n_{k-1} \in N$. For any $n_k \in N$, put

$$H = \prod_{i=1}^{k} k_{n_i}, \quad h_j = \prod_{i \neq j} k_{n_i}, \quad h = \prod_{i=1}^{k} \frac{k_{n_i}}{\sum_{j=1}^{k} h_j} \quad \text{and} \quad \lambda = \frac{k_0^{k-1}}{k_0^{k-1} + 1}.$$

By $\Phi \in \bar{\delta}_2$, there exists $0 < \Theta < 1$ such that

$$\Phi(\lambda u) \leq (1 - \Theta)\lambda\Phi(u), \quad \big(0 \leq u \leq \Phi^{-1}(k_0)\big)$$

(see [2], [3] and [8]). Since Φ is convex, for any $l \in [0, \lambda]$ and $u \in [0, \Phi^{-1}(k_0)]$, we have

$$\Phi(lu) = \Phi\left(\lambda \frac{l}{\lambda} u\right) \leq (1 - \Theta)\lambda\Phi\left(\frac{l}{\lambda} u\right) \leq$$

$$\leq \lambda(1 - \Theta)\frac{l}{\lambda}\Phi(u) \leq (1 - \Theta) l \Phi(u).$$

Since $\dfrac{h_k}{\sum_{i=1}^{k} h_i} = \dfrac{h_k}{h_k + \sum_{i=1}^{k-1} h_i} \leq \dfrac{k_0^{k-1}}{1 + k_0^{k-1}} = \lambda$, there holds

$$\Phi\left(\frac{h_k}{\sum_{i=1}^{k} h_i} u\right) \leq (1 - \Theta)\frac{h_k}{\sum_{i=1}^{k} h_i}\Phi(u) \tag{7}$$

whenever $0 \leq u \leq \Phi^{-1}(k_0)$. By $\Phi \in \delta_2$, there exists $\theta > 0$ such that

$$|I_\Phi(x + y) - I_\Phi(x)| < \frac{\Theta k_0^k}{1 + k_0^{k-1}} \cdot \frac{\delta}{6}.$$

if $I_\Phi(x) \leq k_0$ and $I_\Phi(y) \leq \theta$ (see [2] and [10]).

Notice that $I_\Phi\left(\dfrac{x_{n_1} + x_{n_2} + \cdots + x_{n_{k-1}}}{k}\right) < \infty$ and $I_\Phi(x_{n_i}) < < \infty$ for $i = 1, 2, \ldots, k - 1$. So, there exists $j_0 \in N$ such that

$$\sum_{i=j_0+1}^{\infty} \Phi\left(\frac{x_{n_1}(i) + x_{n_2}(i) + \cdots + x_{n_{k-1}}(i)}{k}\right) < \theta$$

and

$$\sum_{i=j_0+1}^{\infty} \Phi\big(x_{n_j}(i)\big) < \frac{\delta}{3} \quad (j = 1, 2, \ldots, k - 1).$$

By (6), there exists $n_k \in N$ such that

$$\sum_{i=j_0+1}^{\infty} \Phi\big(x_{n_k}(i)\big) \geq \frac{\delta}{3}.$$

Hence

$$\left\|\sum_{i=1}^{k} x_{n_i}\right\|_0 \leq$$

$$\leq \frac{\sum_{i=1}^{k} h_i}{H}\left[1 + I_\Phi\left(\frac{H}{\sum_{i=1}^{k} h_i}(x_{n_1} + x_{n_2} + \cdots + x_{n_{k-1}} + x_{n_k})\right)\right] =$$

$$= \frac{\sum_{i=1}^{k} h_i}{H}\left[1 + \sum_{i=1}^{j_0} \Phi\left(\frac{H}{\sum_{i=1}^{k} h_i}\big(x_{n_1}(i) + \cdots + x_{n_k}(i)\big)\right) + \right.$$

$$\left. + \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{H}{\sum_{i=1}^{k} h_i}\big(x_{n_1}(i) + \cdots + x_{n_k}(i)\big)\right)\right] =$$

$$= \frac{\sum_{i=1}^{k} h_i}{H}\left[1 + \sum_{i=1}^{j_0} \Phi\left(\frac{h_1}{\sum_{i=1}^{k} h_i} k_{n_1} x_{n_1}(i) + \cdots + \frac{h_k}{\sum_{i=1}^{k} h_i} k_{n_k} x_{n_k}(i)\right) + \right.$$

$$+ \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{H}{\sum_{i=1}^{k} h_i} \left(x_{n_1}(i) + \cdots + x_{n_{k-1}}(i)\right) + \frac{H}{\sum_{i=1}^{k} h_i} x_{n_k}(i) \right) \Bigg] \leq$$

$$\leq \frac{\sum_{i=1}^{k} h_i}{H} \left[1 + \sum_{i=1}^{j_0} \left(\frac{h_1}{\sum_{i=1}^{k} h_i} \Phi\left(k_{n_1} x_{n_1}(i)\right) + \cdots + \frac{h_k}{\sum_{i=1}^{k} h_i} \Phi\left(k_{n_k} x_{n_k}(i)\right) \right) + \right.$$

$$\left. + \sum_{i=j_0+1}^{\infty} \Phi\left(\frac{h_k}{\sum_{i=1}^{h} h_i} k_{n_k} x_{n_k}(i) \right) + \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} \right] \leq$$

$$\leq \frac{\sum_{i=1}^{k} h_i}{H} \left[1 + \sum_{i=1}^{j_0} \left(\frac{h_1}{\sum_{i=1}^{k} h_i} \Phi\left(k_{n_1} x_{n_1}(i)\right) + \cdots + \frac{h_k}{\sum_{i=1}^{k} h_i} \Phi\left(k_{n_k} x_{n_k}(i)\right) \right) + \right.$$

$$\left. + (1 - \Theta) \frac{h_k}{\sum_{i=1}^{k} h_i} \sum_{i=j_0+1}^{\infty} \Phi\left(k_{n_k} x_{n_k}(i)\right) + \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} \right] \leq$$

$$\leq \sum_{i=1}^{k} \frac{1}{k_{n_i}} \left(1 + I_\Phi(k_{n_i} x_{n_i})\right) - \Theta \frac{h_k}{\sum_{i=1}^{k} h_i} \sum_{i=j_0+1}^{\infty} \Phi\left(k_{n_k} x_{n_k}(i)\right) +$$

$$+ \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} \leq k - \frac{\Theta k_0^{k-1}}{1 + k_0^{k-1}} \sum_{i=j_0+1}^{\infty} \Phi\left(k_0 x_{n_k}(i)\right) + \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} \leq$$

$$\leq k - \frac{\Theta k_0^k}{1 + k_0^{k-1}} \sum_{i=j_0+1}^{\infty} \Phi\left(x_{n_k}(i)\right) + \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} \leq$$

$$\leq k - \frac{\Theta k_0^k}{1 + k_0^{k-1}} \frac{\delta}{3} + \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})} = k - \frac{\delta \Theta k_0^k}{6(1 + k_0^{k-1})}$$

This completes the proof. □

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