SOME RESULTS CONCERNING POLYMEASURES[1]

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ABSTRACT

We present some results concerning the general theory of polymeasures. Among them, we point out an example of a polymeasure of bounded semivariation and unbounded variation, and two different characterizations of uniform polymeasures.

INTRODUCTION

In this paper we present some results concerning the general theory of *polymeasures,* set functions defined on the product of k algebras which are separately measures.

The case of *bimeasures* ($k = 2$) has been studied by different authors since a long time (see specially [12] and the bibliography there mentioned). Its natural generalization, polymeasures, were introduced by Dobrakov in [7]; in a series of papers (see [8] and the bibliography there mentioned) he developed a very general theory of integration for them. Polymeasures have also been used in, for example, [2], [3], [9], [10] and [11]. We must mention here that, for several results, the case of $k = 2$ is essentially simpler than the case of $k > 2$.

In the first section we present the definitions and some general results. Example 1.4 is interesting because it exhibits a scalar polymeasure of bounded semivariation and unbounded variation.

In the second section we show some results referred to *uniform* polymeasures. We remark that Theorem 2.5 presents a generalization to this kind of polymeasures of a well known theorem of Pettis about measures, and shows that the restriction to *uniform* polymeasures can not be dispensed with.

We present first our notation. Along the paper, k will be a fixed natural number. We will call $\mathcal{A}$ (respectively Σ) to an algebra (respectively σ-algebra) defined on a set K.

We will say that a sequence (A_n) of subsets of K converges to $A \subset K$, and we will write it $A_n \to A$, if $\chi_{A_n}(t) \to \chi_A(t)$ for every $t \in K$. If (A_n) is an increasing (respectively decreasing) sequence, we will write this as $A_n \nearrow A$ (respectively $An \searrow A$). In the sequel Y will denote a Banach space.

1. POLYMEASURES

Definition 1.1. [7, Definition 1] A *function* $\gamma : \mathcal{A}_1 \times \times \cdots \times \mathcal{A}_k \to Y$ *or* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$ *is a (countably additive) k-polymeasure if it is separately (countably) additive.*

Although we present the definition of polymeasures as separately finitely additive set functions defined on the product of algebras, until now only the case of countably additive polymeasures defined on the product of σ-algebras has been studied. For this case we want to mention that if $\Sigma_1 \times \cdots \times \Sigma_k$ are σ-algebras and μ is a countably additive measure defined on $\Sigma_1 \otimes \cdots \otimes \Sigma_k$, the product σ-algebra, then we can obviously define a countably additive polymeasure γ on $\Sigma_1 \times \cdots \times \Sigma_k$ by $\gamma(A_1, ..., A_k) = = \mu(A_1 \times \cdots \times A_k)$. The converse is far from true: let us call $r(\Sigma_1 \times \cdots \times \Sigma_k)$ to the ring generated by $\Sigma_1 \times \cdots \times \Sigma_k$. Then, given a countably additive polymeasure γ defined on $\Sigma_1 \times \cdots \times \Sigma_k$, we can always extend it to a finitely additive measure μ defined on $r(\Sigma_1 \times \cdots \times \Sigma_k)$, but μ need not be countably additive, and it can not be extended in general to $\Sigma_1 \otimes \cdots \otimes \Sigma_k$ (see [6]). In [4], some light is thrown on the question of when a polymeasure can be extended to a measure on the product σ-algebra.

We present now some basic definitions.

Definition 1.2. [7, Definitions 2 and 3] *Given a polymeasure* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$, *its* variation

$$v(\gamma) : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$$

is given by

$$v(\gamma)(A_1, \ldots, A_k) = \sup \left\{ \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} \|\gamma(A_1^{j_1}, \ldots, A_k^{j_k})\| \right\}$$

where the supremum is taken over all the finite $\mathcal{A}_i$-partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i ($1 \le i \le k$).

Likewise we define its supremation

$$\bar{\gamma} : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$$

by

$$\bar{\gamma}(A_1, \ldots, A_k) = \sup \{\|\gamma(B_1, \ldots, B_k)\|\}$$

where the supremum is taken over all the sets $B_i \in \mathcal{A}_i$ such that $B_i \subset A_i$ ($1 \le i \le k$).

And we can define also its semivariation

$$\|\gamma\| : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$$

by

$$\|\gamma\|(A_1, \ldots, A_k) = \sup \left\{ \left\| \sum_{j_1=1}^{n_1} \cdots \sum_{j_k=1}^{n_k} a_1^{j_1} \ldots a_k^{j_k} \gamma(A_1^{j_1}, \ldots, A_k^{j_k}) \right\| \right\}$$

where the supremun is taken over all the finite $\mathcal{A}_i$-partitions $(A_i^{j_i})_{j_i=1}^{n_i}$ of A_i ($1 \le i \le k$), and all the collections $(a_i^{j_i})_{j_i=1}^{n_i}$ contained in the unit ball of the scalar field.

It is clear that these definitions extend the corresponding notions about measures. As in the case of measures, we get that, for every $(A_1, \ldots, A_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ and for every $i \in \{1, \ldots, k\}$,

$$\|\gamma(A_1, \ldots, A_k)\| \le \bar{\gamma}(A_1, \ldots, A_k) \le \|\gamma\|(A_1, \ldots, A_k) \le$$
$$\le v(\gamma)(A_1, \ldots, A_k).$$

$\|\gamma\|, \bar{\gamma}, v(\gamma)$, are separately monotone.

$$\|\gamma\|(A_1, \ldots, A_{i-1}, \emptyset, A_{i+1}, \ldots, A_k) =$$
$$= \bar{\gamma}(A_1, \ldots, A_{i-1}, \emptyset, A_{i+1}, \ldots, A_k) =$$
$$= v(\gamma)(A_1, \ldots, A_{i-1}, \emptyset, A_{i+1}, \ldots, A_k) = 0.$$

The following proposition can be found in [7, Theorem 3].

Proposition 1.3. *Let $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$ be a k-polymeasure. Then, for every $(A_1, \ldots, A_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$,*

$$\bar{\gamma}(A_1, \ldots, A_k) \le \|\gamma\|(A_1, \ldots, A_k) \le 4^k \bar{\gamma}(A_1, \ldots, Ak).$$

From here, it can be extended from the case $k = 1$ the fact that, if $\Sigma_1, \ldots, \Sigma_k$ are σ-algebras defined on sets

$K_1, \ldots, K_k$ and $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \to Y$ is a countably additive polymeasure, then $\|\gamma\|(K_1, \ldots, K_k) < +\infty$ (see [7, Theorems 2 and 3]).

For scalar measures, the variation and semivariation coincide, and this fact is used, more or less explicitly, in the proof of many results concerning both scalar and vector valued measures. The following example shows that the same equality does not hold for scalar polymeasures, and this is in the root of several of the differences that appear between the theory of measures and the theory of polymeasures.

Example 1.4. Let $\Sigma_1 = \mathcal{P}(\mathbb{N})$ and Σ_2 be the Borel sets of $[0, 1]$, let (r_n) be the Rademacher functions and let γ : $\Sigma_1 \times \Sigma_2 \to L^2[0, 1]$ be the countably additive bimeasure ([7, pg. 489]) given by $\gamma(A, B) = P_A(\chi_B)$ where P_A is the ortogonal projection on the subspace $\overline{[(r_n)_{n \in A}]} \subset L^2[0, 1]$. Let $x' = \sum_{n=1}^{\infty} = \dfrac{r_n}{n} \in L^2[0, 1]^*$. Let us define

$$\gamma_{x'} : \Sigma_1 \times E_2 \to \mathbb{R}$$
$$(A, B) \to x' \circ \gamma(A, B)$$

It is easy to see that

$$\gamma_{x'}(A, B) = \sum_{n \in A} \frac{1}{n} \int_B r_n \, d\lambda$$

where λ is the Lebesgue measure on $[0, 1]$. The scalar polymeasure $\gamma_{x'}$ is countably additive and, therefore, it has bounded semivariation (see the comments preceding this example). But we will see now that $v(\gamma)(\mathbb{N}, [0, 1]) = = +\infty$. Let us choose $\{\{1\}, \ldots, \{p\}\}$ as finite partition of $A_p = \{1, \ldots, p\} \subset \mathbb{N}$ and, calling $B_j = \left[\dfrac{j-1}{2^p}, \dfrac{j}{2^p} \right)$, let us choose $(B_j)_{j=1}^{2^p}$ as finite partition of $[0, 1]$. Then

$$v(\gamma_{x'})(\mathbb{N}, [0, 1]) \ge v(\gamma_{x'})(A_p, [0, 1]) \ge \sum_{i=1}^{p} \sum_{j=1}^{2^p} |\gamma_{x'}(\{i\}, B_j)| =$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{2^p} \frac{1}{i} \frac{1}{2^p} = \sum_{i=1}^{p} \frac{1}{i}.$$

(In the previous equalities we use that, for every $i \le p$,

$$\left| \int_{B_j} r_i \, d\lambda \right| = \frac{1}{2^p}.$$

The reason for the variation and the semivariation not to coincide is that the scalars $a_1^{j_1} \ldots a_k^{j_k}$ used in the definition of the latter are not «free enough» to change. This justifies the following definition.

Definition 1.5. *Given a polymeasure* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ $\to Y$, *we define its* quasivariation

$$\|\gamma\|^+ : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$$

by

$$\|\gamma\|^+(A_1, ..., A_k) =$$

$$= \sup \left\{ \left\| \sum_{j_1 = 1}^{n_1} \cdots \sum_{j_k = 1}^{n_k} a_{j_1, ..., j_k} \gamma(A_1^{j_1}, ..., A_k^{j_k}) \right\| \right\}$$

where the supremun is taken over all the finite $\mathcal{A}_i$-*partitions* $(A_i^{j_i})_{j_i = 1}^{n_i}$ *of* $A_i (1 \le i \le k)$, *and all the collections* $(a_{j_1, ..., j_k})$ *contained in the unit ball of the scalar field.*

It is now easy to see that the quasivariation is also separately monotone and that, for every $(A_1, ..., A_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$,

$$\|\gamma\|(A_1, ..., A_k) \le \|\gamma\|^+(A_1, ..., A_k) \le v(\gamma)(A_1, ..., A_k).$$

It is also easy to check that, for scalar polymeasures, the quasivariation and the variation coincide. This quasivariation plays a meaningful role in the theory of polymeasures, as shown in [4] and [6].

The space of polymeasures of bounded semivariation from $\mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ into Y is a Banach space with the semivariation norm. Clearly, the supremation is an equivalent norm for this space. Also, the space of polymeasures of bounded variation is a Banach space with the variation norm and the same happens with the quasivariation.

The following two propositions extend useful well known results concerning measures (see, for example, [1, Proposition I.1.4 and I.1.6]). Their proofs are similar to the proofs of the corresponding results for measures.

Proposition 1.6. *Let* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$ *be a* k-*polymeasure. Let* $D \subset Y^*$ *be a norming subset. Then, for every* $(A_1, ..., A_k) \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$,

$$\|\gamma\|(A_1, ..., A_k) = \sup \{\|y^* \circ \gamma\|(A_1, ..., A_k);$$
$$\|y^*\| \le 1, y^* \in D\}$$

and

$$\|\gamma\|^+(A_1, ..., A_k) = \sup \{v(y^* \circ \gamma)(A_1, ..., A_k);$$
$$\|y^*\| \le 1, y^* \in D\}.$$

Proposition 1.7. *Let* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$ *be a* polymeasure. Then:

a) $v(\gamma) : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to [0, +\infty]$ *is a* k-*polymeasure.*

If γ *is countably additive, then so is* $v(\gamma)$. *Conversely, if* $v(\gamma) < +\infty$ *and* $v(\gamma)$ *is countably additive, then* γ *is countably additive.*

b) $v(\gamma)$ *is the smallest among the positive* k-*polymeasures* λ *which verify*

$$\|\gamma(A_1, ..., A_k)\| \le \lambda(A_1, ..., A_k) \text{ for all } (A_1, ..., A_k) \in$$

$$\in \mathcal{A}_1 \times \cdots \times \mathcal{A}_k.$$

c) $\|\gamma\|$ *and* $\|\gamma\|^+$ *are separately subadditive. Besides, if* γ *is countably additive then* $\|\gamma\|$ *and* $\|\gamma\|^+$ *are separately countably subadditive.*

Corollary 1.8. *Let* γ *be a* k-*polymeasure, then* $\|\gamma\|$ *is a* k-*polymeasure if and only if* $\|\gamma\| = v(\gamma)$. *In particular, if* $v(\gamma)$ *is not finite, then* $\|\gamma\|$ *can not be additive. The same can be said about* $\|\gamma\|^+$.

It is known that extensions of the important Vitali-Hahn-Saks-Nikodým Theorem and Nikodým Theorem are true for polymeasures ([7]). We mention here that this also happens with the Dieudonné-Grothendieck Theorem. The proof can be easily adapted from the proof of the Dieudonné-Grothendieck Theorem for measures (see [5, Corollary I.3.3]), using the Nikodým Theorem for polymeasures when needed.

Proposition 1.9. *(Dieudonné-Grothendieck) Let* Y *be a Banach space and* $D \subset Y^*$ *a norming subset. Let* $\gamma :$ $\Sigma_1 \times \cdots \times \Sigma_k \to Y$ *be an application such that* $(y^* \circ \gamma)$ *is a bounded polymeasure for every* $y^* \in D$. *Then* γ *is a bounded polymeasure.*

2. UNIFORM POLYMEASURES

In general, it is not known whether a countably additive polymeasure has a «control polymeasure» in some reasonable sense. There is a special class of polymeasures, the *uniform* polymeasures, which are «separately uniformly countably additive» in a sense which we will precise in Definition 2.1. These polymeasures have already been considered in [7] and [3]. The main result in this section shows that uniform polymeasures can be characterized as those which are «controlled» by a product measure.

Following the usual notation, we call $ca(\mathcal{A}; Y)$ the set of Y-valued countably additive measures defined on an algebra $\mathcal{A}$.

Definition 2.1. ([7, Definition 1]) *A countably additive polymeasure*

$$\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$$

it is said to be uniform *in the* i^{th}-*variable if the measures*

$$\{\gamma(A_1, ..., A_{i-1}, \cdot, A_{i+1}, ..., A_k) \in$$

$$\in ca(\mathcal{A}_i; Y); (A_1, \overset{[i]}{...}, A_k) \in \mathcal{A}_1 \times \overset{[i]}{...} \times \mathcal{A}_k\}$$

are uniformly countably additive.

A countably additive polymeasure it is said to be uniform *if it is uniform in every variable.*

A measure $\mu : \mathcal{A} \to Y$ is said to be exhaustive (or strongly additive) if, for any sequence of disjoint sets $(A_n) \subset \mathcal{A}$,

$$\lim_{n \to \infty} \|\mu(A_n)\| = 0.$$

As follows from [12, Theorem 4.4], every scalar bimeasure is uniform. This is not true anymore for scalar k-polymeasures, when $k > 2$, as an example in [3] shows. It is also not true for vector valued bimeasures ([7, p. 489]).

Lemma 2.2. *If γ is uniform, then $\bar{\gamma}$ is uniformly separately exhaustive, i.e., if $(A_i^n)_n \subset \mathcal{A}_i$ is a sequence of disjoint sets, then*

$$\lim_{n \to \infty} \sup_{(A_1, \overset{[i]}{...}, A_k) \in \mathcal{A}_1 \times \overset{[i]}{...} \times \mathcal{A}_k} \bar{\gamma}(A_1, ..., A_i^n, ..., A_k) = 0.$$

Proof. Without loss of generality, we will suppose that $i = 1$. Let $(A_1^n)_n \subset \mathcal{A}_1$ be a sequence of disjoint sets. If $\bar{\gamma}$ is not uniformly separately exhaustive, then (A_1^n) has a subsequence which we will also denote (A_1^n) and there exist $\varepsilon > 0$ and sequences $(A_1'^n)_n \subset \mathcal{A}_i$, $(A_2^n) \subset \mathcal{A}_2$, ..., $(A_k^n) \subset \mathcal{A}_k$, such that $A_1'^n \subset A_1^n$ for each $n \in \mathbb{N}$ and such that

$$\|\gamma(A_1'^n, A_2^n, ..., A_k^n)\| > \varepsilon$$

which is a contradiction with γ being uniform. $\square$

We say that a set function μ defined in an algebra $\mathcal{A}$ verifies the Fatou property if for every non decreasing sequence $(A_n) \subset \mathcal{A}$ verifying $A_n \nearrow A \in \mathcal{A}$, we have that $\mu(A_n) \to \mu(A)$.

Theorem 2.3. *Let $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$ be a countably additive k-polymeasure. Then the following are equivalent.*

i) γ *is uniform.*

ii) $\bar{\gamma}$ *is uniformly separately continuous in $\emptyset$, i.e., if $(A_i^n)_n \subset \mathcal{A}_i$ verifies that $A_i^n \searrow \emptyset$ then we have that*

$$\lim_{n \to \infty} \sup_{(A_1, \overset{[i]}{...}, A_k) \in \mathcal{A}_1 \times \overset{[i]}{...} \times \mathcal{A}_k} \|\gamma\|(A_1, ..., A_i^n, ..., A_k) = 0.$$

iii) $\bar{\gamma}$ *is separately continuous in $\emptyset$.*

Proof. (i) $\Rightarrow$ (ii): [7, Theorem 2] states that $\bar{\gamma}$ is always separately monotone and it verifies separately the Fatou property. Let us suppose that $i = 1$. If $\bar{\gamma}$ is not uniformly countably additive in $\emptyset$, then there exist

$(A_1^n, ..., A_k^n)_n \subset \mathcal{A}_1 \times \cdots \times \mathcal{A}_k$ and $\varepsilon > 0$ such that $A_1^n \searrow \emptyset$ and, for every $n \in \mathbb{N}$,

$$\bar{\gamma}(A_1^n, A_2^n, ..., A_k^n) > \varepsilon.$$

Let $B_1^n = A_1^n \backslash A_1^{n+1}$ and let $C_n^m = \bigcup_{j=n}^m B_1^j = A_1^n \backslash A_1^m$. Since $C_n^m \nearrow A_1^n$ when m grows to infinite and since $\bar{\gamma}$ verifies separately the Fatou property, we get that, when m grows to infinite

$$\bar{\gamma}(C_n^m, A_2^n, ..., A_k^n) \nearrow \bar{\gamma}(A_1^n, A_2^n, ..., A_k^n) > \varepsilon.$$

So, for $n_1 = 1$, there exists m_1 such that

$$\bar{\gamma}(C_n^{m_1}, A_2^{n_1}, ..., A_k^{n_1}) > \frac{\varepsilon}{2}.$$

Let now $n_2 = m_1 + 1$. There exists m_2 such that

$$\bar{\gamma}(C_{n_2}^{m_2}, A_2^{n_2}, ..., A_k^{n_2}) > \frac{\varepsilon}{2}.$$

Continuing with this procedure we obtain two intertwined sequences $1 = n_1 < m_1 < n_2 < m_2...$ such that, for every $l \in \mathbb{N}$,

$$\bar{\gamma}(C_{n_l}^{m_l}, A_2^{n_l}, ..., A_k^{n_l}) > \frac{\varepsilon}{2}$$

which contradicts the previous lemma.

ii) implies iii) is obvious.

Let us now suppose that $\bar{\gamma}$ is separately continuous in $\emptyset$. If γ is not uniform in, for example, the first variable, then there exist $\varepsilon > 0$, a sequence $(A_1^n)_n \subset \mathcal{A}_1$ of disjoint sets, an increasing sequence of indices $(n(m))_m$ and a sequence $(A_2^m, ..., A_k^m)_m \subset \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ such that, for every $m \in \mathbb{N}$,

$$\left\| \gamma\left(\bigcup_{n=1}^{\infty} A_1^n, A_2^m, ..., A_k^m\right) - \sum_{n=1}^{m} \gamma(A_1^n, A_2^m, ..., A_k^m) \right\| > \varepsilon$$

and from here it follows that

$$\left\| \gamma\left(\bigcup_{n=m}^{\infty} A_1^n, A_2^m, ..., A_k^m\right) \right\| > \varepsilon.$$

Then, taking $A_i = \bigcup_{m=1}^{\infty} A_i^m$ ($2 \leq i \leq k$), we obtain that $\bigcup_{n=m}^{\infty} A_1^n \searrow \emptyset$ as m grows to infinite, but

$$\bar{\gamma}\left(\bigcup_{n=m}^{\infty} A_1^n, A_2, ..., A_k\right) > \varepsilon \text{ for every } m \in \mathbb{N},$$

a contradiction with the fact that $\bar{\gamma}$ be separately continuous in $\emptyset$. $\square$

If A and B are two subsets of a set K, their symmetric difference, $A \triangle B$, is defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Corollary 2.4. [7, Theorem 7] *Let* $\gamma : \mathcal{A}_1 \times \cdots \times A_k \to Y$ *be a uniform polymeasure and let* $(A_i^{n_i})_{n_i} \subset A_i$ $(1 \leq i \leq k)$, *be sequences such that* $(A_i^{n_i}) \to A_i \in \mathcal{A}_i$. *Then*

$$\lim_{n_1, \ldots, n_k \to \infty} \overline{\gamma}(A_1^{n_1}, \ldots, A_k^{n_k}) = \overline{\gamma}(A_1, \ldots, A_k).$$

Proof. Let us observe that

$$\|\overline{\gamma}(A_1^{n_1}, \ldots, A_k^{n_k}) - \overline{\gamma}(A_1, \ldots, A_k)\| \leq$$

$$\leq \|\overline{\gamma}(A_1^{n_1}, \ldots, A_k^{n_k}) - \overline{\gamma}(A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}, A_k)\| + \cdots +$$

$$+ \cdots + \|\overline{\gamma}(A_1^{n_1}, A_2, \ldots, A_k) - \overline{\gamma}(A_1, \ldots, A_k)\|.$$

Now, we observe that $\overline{\gamma}$ is monotone and subadditive and that every positive monotone and subadditive set function verifies that $|v(A) - v(B)| \leq 2|v(A \triangle B)|$ for every A and B in $\mathcal{A}$ ([7, pg. 500]). With this we obtain that

$$\|\overline{\gamma}(A_1^{n_1}, \ldots, A_k^{n_k}) - \overline{\gamma}(A_1, \ldots, A_k)\| \leq$$

$$\leq 2\|\overline{\gamma}(A_1^{n_1}, \ldots, A_{k-1}^{n_{k-1}}, A_k \triangle A_k^{n_k})\| + \cdots +$$

$$+ \cdots + \|\gamma(A_1 \triangle A_1^{n_1}, A_2, \ldots, A_k)\|$$

and this converges to 0 because $\overline{\gamma}$ is uniformly separately continuous in $\emptyset$.   □

The following proposition allows us to characterize uniform polymeasures as those which verify certain Pettis-type theorem.

Theorem 2.5. *Let* $\gamma : \Sigma_1 \times \cdots \times \Sigma_k \to Y$ *be a countably additive polymeasure. Then* γ *is uniform if and only if there are k countably additive measures* $\lambda_i : \Sigma_i \to [0, +\infty)$ $(1 \leq i \leq k)$, *such that*

$$\lim_{\lambda_1 \times \cdots \times \lambda_k(A_1, \ldots, A_k) \to 0} \gamma(A_1, \ldots, A_k) = 0$$

Proof. Let us first suppose that γ is uniform. Then [7, Theorem 10] states that there exist k countably additive measures $\lambda_i : \Sigma_i \to [0, +\infty)$ $(1 \leq i \leq k)$, such that $\gamma(A_1, \ldots, A_k) = 0$ when $\lambda_1 \times \cdots \times \lambda_k(A_1, \ldots, A_k) = 0$. If the result is not true, then there exist $(A_1^n, \ldots, A_k^n)_n \subset \Sigma_1 \times \cdots \times \Sigma_k$ and $\varepsilon > 0$ such that

$$\lim_{n \to \infty} (\lambda_1 \times \cdots \times \lambda_k)(A_1^n, \ldots, A_k^n) = 0 \quad \text{and, for every } n \in \mathbb{N},$$

$$\|\gamma(A_1^n, \ldots, A_k^n)\| > \varepsilon.$$

Since the sequences $(\lambda_i(A_i^n))_n$ are bounded, we can, taking k times subsequences if necessary, consider that, for every $i \in \{1, \ldots, k\}$ the sequence $(\lambda_i(A_i^n))_n$ converges. Then

$$0 = \lim_{n \to \infty} (\lambda_1 \times \cdots \times \lambda_k)(A_1^n, \ldots, A_k^n) =$$

$$= \lim_{n \to \infty} \lambda_1(A_1^n) \cdots \lim_{n \to \infty} \lambda_k(A_k^n)$$

and therefore there exists i such that $\lim_{n \to \infty} \lambda_i(A_i^n) = 0$. Let us suppose without loss of generality that this happens for $i = 1$. We can, passing again to a subsequence if necessary, suppose that $\lambda(A_1^n) < \frac{1}{2^n}$. Let $B_m = \cup_{n=m}^{\infty} A_1^n$.

Then $\lambda_1(B_m) \leq \frac{1}{2^{m-1}}$. Since $(B_m)_m$ is not increasing, we obtain that $B_m \to B$ with $\lambda_1(B) = 0$, which by Corollary 2.4, implies that, for every $(A_2, \ldots, A_k) \in \Sigma_2 \times \cdots \times \Sigma_k$,

$$\overline{\gamma}(B, A_2, \ldots, A_k) = 0.$$

So, we get that

$$\varepsilon < \|\gamma(A_1^n, \ldots, A_k^n)\| \leq \overline{\gamma}(A_1^n, \ldots, A_k^n) \leq$$

$$\leq \gamma\left(\bigcup_{m=n}^{\infty} A_1^m, A_2^n, \ldots, A_k^n\right) \leq$$

$$\leq \overline{\gamma}(B, A_2^n, \ldots, A_k^n) + \overline{\gamma}\left(\left(\bigcup_{m=n}^{\infty} A_1^m\right) \setminus B, A_2^n, \ldots, A_k^n\right).$$

The first one of these terms is zero and the second one converges to zero because $\left((\cup_{m=n}^{\infty} A_1^m) \setminus B\right) \searrow \emptyset$, which leads us to a contradiction.

Conversely, let us suppose that there exist $\lambda_i : \Sigma_i \to [0, +\infty)(1 \leq i \leq k)$, as in the hypothesis. If γ is not uniform, then $\overline{\gamma}$ is not separately continuous in $\emptyset$. Then, there exist $(A_1^n)_n \subset \Sigma_1$ such that $A_1^n \searrow \emptyset$, and there exists $(A_2^n, \ldots, A_k^n)_n \subset \Sigma_2 \times \cdots \times \Sigma_k$ and $\varepsilon > 0$ such that, for every $n \in \mathbb{N}$,

$$\overline{\gamma}(A_1^n, \ldots, A_k^n) > \varepsilon.$$

In that case, there exist $B_i^n \subset A_i^n$ $(1 \leq i \leq k, n \in \mathbb{N})$ such that

$$\|\gamma(B_1^n, \ldots, B_k^n)\| > \varepsilon,$$

but, clearly, $B_1^n \searrow \emptyset$ and, therefore, $\lambda_1(B_1^n) \searrow 0$. Since, for every $i \in \{2, \ldots, k\}$, λ_i, is bounded, we get that

$$\lim_{n \to \infty} \lambda_1 \times \cdots \times \lambda_k(B_1^n, \ldots, B_k^n) = 0,$$

a contradiction.   □

Our last proposition gives a sufficient condition for a polymeasure to be uniform.

Proposition 2.6. *Let* $\gamma : \mathcal{A}_1 \times \cdots \times \mathcal{A}_k \to Y$ *be a countably additive polymeasure. If γ has bounded variation, then γ is uniform.*

Proof. If γ has bounded variation, then $v(\gamma)$ is a countably additive polymeasure with values in a Banach space (see Proposition 1.7). From [7, Theorem 1] it follows

that $v(\gamma)$, and therefore $\bar{\gamma}$, are separately continuous in $\emptyset$; hence, applying Theorem 2.3, we get that γ is uniform. $\square$

The converse is not true: obviously, every countably additive measure is uniform. Also, Example 1.4 provides a scalar bimeasure (and thus uniform, see the comments following Definition 2.1) of unbounded variation.

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