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ON THE LIMITATIONS OF THE GROTHENDIECK TECHNIQUES

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ABSTRACT

Let T be a locally compact Hausdorff space, and let $C_0(T)$ be the Banach space of all complex valued continuous functions f vanishing at infinity in T with $\|f\|_T = \sup_{t\in T}|f(t)|$. The aim of the present note is to show that the Grothendieck techniques are not powerful enough to prove the Dieudonné property of $C_0(T)$ if T is an arbitrary locally compact Hausdorff space. In fact, his method of proof is valid if and only if T is further σ-compact. However, one can prove the Dieudonné property of $C_o(T)$ for arbitrary T by appealing to the results of an earlier article of the author (see Remarks 3 below).

1. INTRODUCTION

Let T be a locally compact Hausdorff space. Let X be a locally convex Hausdorff space (briefly, a lcHs) which is quasicomplete. Let $C_o(T)$ be the Banach space of all complex valued continuous functions f vanishing at infinity in T with $\|f\|_T = \sup_{t\in T}|f(t)|$. $M(T)$ denotes the Banach dual of $C_o(T)$ and consists of all bounded complex Radon measures on T.

Grothendieck proved in [2] that $C_o(T)$ has the strict Dunford-Pettis property (see Theorem 1 of [2]). Theorem 3 in [2] says that a bounded subset A of $M(T)$ is relatively compact with respect to $\sigma(M(T), M^*(T))$ if and only if it is so with respect to $\sigma(M(T), \beta_0)$, where β_o is the vector subspace of $M^*(T)$ spanned by the characteristic functions of all closed sets in T. At the end of the proof Grothendieck comments in the Remark on p. 152 of [2] that his Theorem 3 continues to be valid if β_o is replaced by the vector subspace of $M^*(T)$ spanned by the characteristic functions of all closed G_δ sets in T and considers, for simplicity, the compact case. For our reference below, we shall call it the strengthened version of Theorem 3 of [2].

Theorem 6 of [2] states that $C(K)$, K a compact Hausdorff space, has Dieudonné property and more precisely, for a continuous linear map $u : C(K) \to X$, where X is a complete lcHs, the following conditions are equivalent:

(i) u is weakly compact.

(ii) For each closed set F in T, $u^{**}(\chi_F) \in X$.

(iii) For each closed G_δ set F in T, $u^{**}(\chi_F) \in X$.

(iv) For each non decreasing bounded sequence (f_n) of positive functions in $C(K)$, $(u(f_n))$ converges weakly in X.

The validity of the Dieudonné property for $C(K)$ is a consequence of the above characterizations of weakly compact operators $u : C_0(T) \to X$. His proof is based on the strict Dunford-Pettis property of $C(K)$, the strengthened version of Theorem 3 of [2] and Proposition 11 of [2].

Then in Remark 2 on p. 161 of [2] Grothendieck comments that with the help of his techniques developed in earlier sections (namely, the strict Deunford-Pettis property of $C_o(T)$, the strengthened version of Theorem 3 of [2] and Proposition 11 of [2]) one can show without much difficulty that the statements of his Theorem 6 are textually valid for $C_o(T)$, when T is a locally compact Hausdorff space.

Later, Edwards carried out the suggestions of Grothendieck and obtained in Theorem 9.10.4 of [1] the locally compact analogue of Theorem 6 of [2], which is the same as Theorem 7 in Section 4 without the σ-compactness of T. His proof of (1) $\Rightarrow$ (3) of the said theorem is incorrect, but, as pointed out in [4], can be rectified by appealing to the strict Dunford-Pettis property of $C_o(T)$. In this note we show that his proofs of (3) $\Rightarrow$ (2 bis) and (2 bis) $\Rightarrow$ (1) of the above theorem are also incorrect without the additional hypothesis of σ-compactness of T. In fact, we establish here that the Grothendieck techniques can be applied to prove the locally compact version of Theorem 6 of [2] if and only if the locally compact space is further σ-compact. In other words, the Grothendieck techniques are not powerful enough to obtain the locally compact analogue of his Theorem 6, contrary to his claim in Remark 2 on p. 161 of [2].

However, using the new techniques developed in [3, 4], the author has obtained in [4] several characterizations for a continuous linear map $u : C_o(T) \to X$ to be weakly compact, where T is an arbitrary locally compact Hausdorff space and X is a quasicomplete lcHs. These characterizations include those mentioned in Remark 2 on p. 161 of Grothendieck [2] or in Theorem 9.4.10 of [1] and as a consequence, $C_o(T)$ has Dieudonné property, though T is not σ-compact.

2. PRELIMINARIES

Let T be a locally compact Hausdorff space and let $C_o(T)$ and $M(T)$ be as in Introduction. Given $\mu \in M(T)$, μ gives rise to a unique regular complex Borel measure on T, which too is denoted by μ. Conversely, given a regular complex Borel measure μ on T, there exists a unique bounded complex Radon measure (which too is denoted by μ) on T to which it corresponds. For this reason, we shall treat $M(T)$ also as the set of all regular complex Borel measures on T.

Definition 1. *Let X be an lcHs. By the first Baire class of X^{**} [which is the dual of $(X^*, \beta(X^*, X))$], we mean the subspace of X^{**} formed by the $\sigma(X^{**}, X^*)$-limits of $\sigma(X, X^*)$-Cauchy sequences of elements in X.*

We slightly modify the second part of Definition 4 of [2] as below.

Definition 2. *Let X be an lcHs and let H be the first Baire class of X^{**}. Then X is said to have Dieudonné property if for each quasicomplete lcHs Y, every continuous linear map $u : X \to Y$ with $u^{**}(H) \subset Y$ satisfies $u^{**}(X^{**}) \subset Y$.*

Lemma 1 of [2] has been strengthened as Corollary 9.3.2 in [1], with the image space just quasicomplete instead of being complete as in [2]. Since every Banach space is a quasicomplete lcHs, and since only Corollary 9.3.2 of [1] is used (instead of Lemma 1 of [2]) in the proof of Proposition 9.4.9 of [1] (which is the same as Proposition 11 of [2]), one can replace the completeness hypothesis of the image space in the said proposition by that of quasicompleteness. With this observation, we modify Proposition 9.4.9 of [1] as below.

Proposition 3. *Let X be an lcHs and let Φ be a family of $\sigma(X^{**}, X^*)$-convergent nets of elements of X. Let H be the vector subspace of X^{**} spanned by X and the limits of members of Φ. Then the following are equivalent:*

(1) *If $u : X \to Y$ is a continuous linear map, with Y a quasicomplete lcHs and $u^{**}(H) \subset Y$, then $u^{**}(X^{**}) \subset Y$.*

(2) *Every equicontinuous, convex, balanced and $\sigma(X^*, H)$-compact set in X^* is also $\sigma(X^*, X^{**})$-compact.*

3. ON THE PROOF OF THEOREM 4.22.3 OF [1]

Let β_o be the vector subspace of $M^*(T)$ spanned by the characteristic functions of all closed sets in T. Let $\hat{T} = T \cup \{\omega\}$ be the Alexandroff compactification of T. It is easy to check by the definition of the topology of $\hat{T}$ that the vector subspace $\hat{\beta}_o$ of $M(\hat{T})$ spanned by the characteristic functions of closed sets in $\hat{T}$ is given by $\hat{\beta}_o = \beta_o \oplus \mathbb{C}\chi_\omega$. Consequently, the argument of reduction to the compact case as given in the proof of Theorem 3 of [2] is valid.

Edwards [1] uses Grothendieck's proof of Theorem 3 of [2] to prove its strengthened version, namely Theorem 4.22.3 of [1] which is the same as Proposition 5 below without the σ-compactness hypothesis of T. As in [2], he identifies $M(T)$ with the closed hyperplane $N = \{\lambda \in M(\hat{T}) : \lambda(\{\omega\}) = 0\}$ and then tacitly assumes, as in the original proof of Theorem 3 of [2], that $(M(T), \sigma(M(T), Q))$ and $(N, \sigma(M(\hat{T}), \hat{Q})|_N)$ are homeomorphic under this identification, where Q (resp. $\hat{Q}$) is the vector subspace of $M^*(T)$ (resp. $M^*(\hat{T})$) spanned by the characteristic functions of all closed G_δ sets in T (resp. in $\hat{T}$). Unlike the case of $\hat{\beta}_o$, the characteristic functions of many closed non compact G_δ sets in T will not belong to $\hat{Q}$ if T is not σ-compact, i.e. if $\{\omega\}$ is not G_δ. Thus one needs a proof to establish the said homeomorphism as it is no longer obvious and this result is essential to justify the argument of reduction to the compact case in the proof of Theorem 4.22.3 of [1].

In this section we prove the homeomorphism of the said spaces under the additional hypothesis that T is σ-compact.

Proposition 4. *Let T be a locally compact Hausdorff space, and let $\hat{T} = T \cup \{\omega\}$ be its Alexandroff compactification. Let Q (resp. $\hat{Q}$) be the vector subspace of $M^*(T)$ (resp. $M^*(\hat{T})$) spanned by the characteristic functions of all closed G_δ sets in T (resp. in $\hat{T}$). Then there is an isometric isomorphism Ψ of $M(T)$ onto the closed subspace $N = \{\lambda \in M(\hat{T}) : |\lambda|(\{\omega\}) = 0\}$ of $M(\hat{T})$. Moreover, if the space T is further σ-compact, then the spaces $(M(T), \sigma(M(T), Q))$ and $(N, \sigma(M(\hat{T}), \hat{Q})|_N)$ are homeomorphic under the map Ψ.*

Proof. Let $\mathcal{B}(T)$ and $\mathcal{B}(\hat{T})$ be the σ-algebras of Borel sets in T and $\hat{T}$, respectively. Then by the definition of the topology of $\hat{T}$ we observe that a subset E of $\hat{T}$ belongs to $\mathcal{B}(\hat{T})$ if and only if $E\backslash\{\omega\} \in \mathcal{B}(T)$. Given $\mu \in M(T)$, let $\Psi(\mu)(E) = \mu(E\backslash\{\omega\})$ for $E \in \mathcal{B}(\hat{T})$. Then clearly $\Psi(\mu)$ is a regular complex Borel measure on $\hat{T}$ and the complex Radon measure $\Psi(\mu)$ determined by it has $\|\Psi(\mu)\| = \|\mu\|$ and $|\Psi(\mu)|(\{\omega\}) = 0$. Thus $\Psi(\mu) \in N$ for $\mu \in M(T)$. Clearly, the above argument is reversible and hence Ψ is an isometric isomorphism of $M(T)$ onto N.

Hereafter we shall assume that T is further σ—compact.

Claim 1. $\hat{Q} = Q \oplus \mathbb{C}\chi_{\omega}$.

In fact, let $\mathcal{F} = \{F \subset T : F$ closed G_δ in $T\}$, $\mathcal{F}_1 = \{F \subset T$: F compact G_δ in $T\}$ and $\mathcal{F}_2 = \{F \subset T : F$ non compact closed G_δ in $T\}$. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $\mathcal{G} = \{\mathcal{G} \subset \hat{T}$: G closed G_δ in $\hat{T}\}$. For each $F \in \mathcal{F}_1$, let $\hat{F} = F$; and for each $F \in \mathcal{F}_2$, let $\hat{F} = F \cup \{\omega\}$. By the definition of the topology of $\hat{T}$, $\hat{\mathcal{F}}_1 = \{\hat{F} : F \in \mathcal{F}_1\} \subset \mathcal{G}$. As T is σ-compact, $\{\omega\}$ is G_δ in $\hat{T}$. Hence there exists a non increasing sequence (V_n) of open sets in $\hat{T}$ such that $\{\omega\} = \cap_1^\infty V_n$. If $F \in \mathcal{F}_2$ then there exists a non increasing sequence (U_n) of open sets in T such that $F = \cap_1^\infty U_n$. Thus, for $F \in \mathcal{F}_2$, we have $\hat{F} = F \cup \{\omega\} = \cap_1^\infty (U_n \cup V_n)$ and hence $\hat{F} \in \mathcal{G}$. Thus $\mathcal{F} \subset \mathcal{G}$. Conversely, let $G \in \mathcal{G}$. Then either $\omega \in G$ or $\omega \notin G$. If $\omega \in G$, let $G = \cap_1^\infty V_n$, V_n open in $\hat{T}$. Then $G\backslash\{\omega\}$ is non compact and closed in T and $G\backslash\{\omega\} = \cap_1^\infty (V_n\backslash\{\omega\})$ with each $V_n\backslash\{\omega\}$ open in T. Thus $G\backslash\{\omega\} \in \mathcal{F}_2$, and if $F = G\backslash\{\omega\}$, then $G = \hat{F}$ so that $G \in \hat{\mathcal{F}}$. If $\omega \notin G$, let $G = \cap_1^\infty V_n$, V_n open in $\hat{T}$. Then $G = \cap_1^\infty (V_n\backslash\{\omega\})$ with $V_n\backslash\{\omega\}$ open in T for each n. Moreover, G is a compact set in T. Thus $G \in \mathcal{F}_1$. Therefore, $\mathcal{G} \subset \hat{\mathcal{F}}$. This proves that $\hat{\mathcal{F}} = \mathcal{G}$. Consequently, $\hat{Q} = Q \oplus \mathbb{C}\chi_{\omega}$.

Note that a net (μ_α) in $M(T)$ converges to $\mu \in M(T)$ with respect to the topology $\sigma(M(T), Q)$ if and only if $\mu_\alpha(f) \to \mu(f)$ for each $f \in Q$, and $\Psi(\mu_\alpha) \to \Psi(\mu)$ in N with respect to the topology $\sigma(M(\hat{T}), \hat{Q})|_N$ if and only if $\Psi(\mu_\alpha)(g) \to \Psi(\mu)(g)$ for each $g \in \hat{Q}$. Now by Claim 1 and by the fact that $\lambda(\{\omega\}) = 0$ for each $\lambda \in N$, we conclude that $\mu_\alpha \to \mu$ in $M(T)$ with respect to $\sigma(M(T), Q)$ if and only if $\Psi(\mu_\alpha) \to \Psi(\mu)$ in N with respect to $\sigma(M(\hat{T}), \hat{Q})|_N$.

This completes the proof of the proposition.

Thus, under the additional hypothesis of σ-compactness of T, Proposition 4 justifies the argument of reduction to the compact case in the proof of Theorem 4.22.3 of [1]. Since the proof in the remaining part as given in [1] holds, we have the following proposition (modified version of Theorem 4.22.3 of [1]).

Proposition 5. *Let T be a σ-compact locally compact Hausdorff space, and let Q be the vector subspace of $M^*(T)$ spanned by the characteristic functions of all closed G_δ sets in T. Then a bounded set A in $M(T)$ is relatively compact with respect to $\sigma(M(T), M^*(T))$ if and only if it is so with respect to $\sigma(M(T), Q)$.*

Remarks 1. It seems that the homeomorphism mentioned in Proposition 4 may fail without the hypothesis of σ-compactness of T. If it fails, then the validity of Theorem 4.22.3 of [1] (for arbitrary locally compact Hausdorff spaces) remains to be settled.

4. DIEUDONNÉ PROPERTY OF $C_o(T)$, T σ-COMPACT

We shall show in this section that the Grothendieck techniques mentioned in Introduction can be applied to prove the locally compact version of Theorem 6 of [2] if and only if the locally compact space is futher σ-compact. Let us begin with the following proposition.

Proposition 6. *For each open F_σ set U in the locally compact Hausdorff space T there exists a non decreasing sequence (f_n) of positive functions in $C_o(T)$ with $f_n \nearrow \chi_U$ if and only if T is σ-compact.*

Proof. Let T be σ-compact. If the open set U is F_σ, then clearly U is σ-compact. Let $U = \cup_1^\infty K_n$ with K_n compact for each n. Since $K_n \subset U$, K_n is compact and U is open, by Urysohn's lemma there exists a $g_n \in C_c(T)$ with compact support contained in U such that $0 \leq g_n \leq 1$ in T and $g_n(t) = 1$ for $t \in K_n$. Let $f_n = \max_{1 \leq k \leq n} g_k$. Then $(f_n)_1^\infty \subset C_o(T)$ and $f_n \nearrow \chi_U$.

Conversely, as T is an F_σ open set, by hypothesis there exists a sequence of positive functions $(f_n)_1^\infty \subset C_o(T)$ such that $f_n \nearrow \chi_T$. Given $n, k \in \mathbb{N}$, there exists a compact $K_{n,k}$ in T such that $|f_n(t)| < \frac{1}{k}$ for all $t \in T\backslash K_{n,k}$. If $U_n = \{t : f_n(t) > 0\}$, then U_n is open and $U_n = \cup_{k=1}^\infty \left\{t : f_n(t) \geq \frac{1}{k}\right\}$. Let $F_{n,k} = \left\{t : f_n(t) \geq \frac{1}{k}\right\}$. Then $U_n \nearrow T$ and $T = \cup_1^\infty U_n = \cup_{n=1}^\infty \cup_{k=1}^\infty F_{n,k} \subset \cup_{n=1}^\infty \cup_{k=1}^\infty K_{n,k} \subset T$. Thus T is σ-compact.

This completes the proof of the proposition.

Corollary. *The characteristic functions χ_U of open F_σ sets U (resp. χ_F of closed G_δ sets F) in T are pointwise limits of non decreasing (resp. non increasing) sequences of positive functions in $C_o(T)$ if and only if T is σ-compact.*

Using the Grothendieck techniques we prove below the locally compact analogue of Theorem 6 of [2] under the additional hypothesis that the locally compact space is σ-compact. See Remarks 2 for the necessity of the hypothesis of σ-compactness to apply the Grothendieck techniques.

Theorem 7. *Let T be a σ-compact locally compact Hausdorff space. Then $C_o(T)$ has Dieudonné property. More precisely, given a continuous linear map $u : C_o(T) \to X$, where X is a quasicomplete lcHs, the following conditions are equivalent:*

(1) *u is weakly compact.*

(2) *For each closed set F in T, $u^{**}(\chi_F) \in X$.*

(3) *For each closed $G\delta$ set F in T, $u^{**}(\chi F) \in X$.*

(4) *For each non decreasing bounded sequence (f_n) of positive functions in $C_o(T)$, $(u(f_n))$ converges weakly in X.*

Proof.

(1) $\Rightarrow$ (2) by Corollary 9.3.2 of [1] or by Lemma 1 of [2].

(2) $\Rightarrow$ (3). It is obvious.

(1) $\Rightarrow$ (4). Such a sequence (f_n) is weakly Cauchy by the Lebesgue bounded convergence theorem and consequently, by the strict Dunford-Pettis property of $C_o(T)$, the sequence $(u(f_n))$ converges in the topology of X. Thus, in particular, (4) holds.

(4) $\Rightarrow$ (3). Obviously, it suffices to show that $u^{**}(\chi_U) \in X$ for each open F_σ set U in T. Let U be such a set in T. As T is σ-compact, then by Proposition 6 there exists a non decreasing sequence (f_n) of positive functions in $C_o(T)$ such that $f_n \nearrow \chi_U$. Then by hypothesis (4), there exists a vector $x_o \in X$ such that $u(f_n) \to x_o$ weakly. As $u^* : X^* \to M(T)$, by the Lebesgue bounded convergence theorem we have

$$\langle x_o, x^* \rangle = \lim_n \langle u(f_n), x^* \rangle = \lim_n \langle f_n, u^*x^* \rangle = \langle \chi_U, u^*x^* \rangle$$

and thus

$$\langle x_o, x^* \rangle = \langle u^{**}(\chi_U), x^* \rangle$$

for each $x^* \in X^*$. Therefore, $u^{**}(\chi_U) = x_o \in X$. Hence (3) holds.

(3) $\Rightarrow$ (1). Let Q be the vector subspace of $C_o^{**}(T)$ spanned by the characteristic functions χ_F of closed G_δ sets F in T. Then, as T is σ-compact, by Corollary to Proposition 6 there exists a non increasing sequence (f_n) of positive functions in $C_o(T)$ such that $fn \searrow \chi_F$, for each closed G_δ set F in T. Let $\Phi = \{(f_n) \subset C_o(T) : f_n \searrow \chi_F,\ F$ closed G_δ in $T\}$. Then by the Lebesgue bounded convergence theorem, (f_n) is $\sigma(C_0^{**}(T), M(T))$-convergent in $C_0^{**}(T)$ for each $(f_n) \in \Phi$. Let H be the vector subspace of $C_o^{**}(T)$ spanned by $C_o(T)$ and the limits of members of Φ. Then $Q \subset H$. Now by hypothesis (3), by Propositions 3 and 5 above and by Corollary 9.3.2 of [1] we conclude that u *is* weakly compact. Hence (1) holds.

This completes the proof of the theorem.

Remarks 2. The hypothesis that T is σ-compact is essential in the above proof of (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1), as Proposition 6 and its Corollary are used. If T is not σ-compact, then by Corollary to Proposition 6, χ_T and the characteristic functions of many closed G_δ sets in T are no longer the limits of non increasing sequences of positive functions in $C_o(T)$ and hence neither (4) implies (3) nor (3) implies (1). In other words, the Grothendieck techniques are applicable if and only if T is further σ-compact.

Remarks 3. Using the techniques developed in [3, 4], the author has obtained in [4] 35 characterizations for a continuous linear map $u : C_o(T) \to X$ to be weakly compact, where T is an arbitrary locally compact Hausdorff space and X is a quasicomplete lcHs. Since these characterizations include those of Theorem 7 above, $C_o(T)$ has Dieudonné property even though T is not σ-compact. In this connection, the reader may refer to [5], where the author has obtained the said characterizations by using the regular Borel extension of X-valued Baire measures on T.

Remarks 4. Even if Theorem 4.22.3 of [1] were true for arbitrary locally compact Hausdorff spaces T, as observed in Remarks 2, the hypothesis of σ-compactness of T cannot be dispensed with in Theorem 7 (if the Grothendieck techniques are to be employed).

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