Rev.R.Acad.Cienc.Exact.Fis.Nat. (Esp)
Vol. 94, N.º 4, pp 431-436, 2000
Monográfico: Perspectivas en Análisis Matemático

GEOMETRIC PROPERTIES RELATED TO THE FIXED POINT PROPERTY IN BANACH SPACES

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ABSTRACT

We investigate the relationship between properties GLD, GGLD, δ, WNS and KK and we prove that c_0 cannot be renormed to have GGLD. Finally we study properties GLD and GGLD in $J(x_i)$ and $JT(x_i)$.

1. INTRODUCTION

In the last few years several geometric properties implying weak normal structure have appeared in the literature, among them those defined by Gossez and Lami Dozo [11], by Jiménez-Melado [12] and by Khamsi [13]. Our aim is to put those properties under perspective and also to study their relationship with other concepts related to these issues, like for instance the Kadec-Klee property and normal structure (see e.g. [10]). We also prove that c_0 cannot be renormed in order to have either the property defined by Jiménez-Melado or that of Gossez and Lami Dozo. In the last section we study those properties in generalized James spaces.

2. DEFINITIONS

We start by giving a list of the definitions of the properties which will be the object of our study.

Definition 2.1. *A Banach space X with a Schauder basis has the Gossez-Lami Dozo property (GLD) if for each $\varepsilon > 0$, there exists $r > 0$ such that for every $x \in X$ and $n \in \mathbb{N}$ we have*

$$\|P_n x\| = 1 \quad \text{and} \quad \|(I - P_n)x\| \geq \varepsilon \text{ imply } \|x\| \geq 1 + r,$$

where $\{P_n\}_n$ is the sequence of natural projections associated with the basis.

In 1992 Jiménez-Melado [12] generalized this concept as follows:

Definition 2.2. *A Banach space X has the generalized Gossez-Lami Dozo property (GGLD) if for every weakly null sequence $\{y_n\}_n$ such that $\lim_n \|y_n\| = 1$ we have that $D\{y_n\} > 1$, where*

$$D\{y_n\} = \limsup_m \left(\limsup_n \|y_n - y_m\|\right).$$

The next definition is a variation of a property defined by Khamsi [13]. It was studied in detail in [7] and [8].

Definition 2.3. *A Banach space X with a Schauder basis $\{x_n\}_n$ is said to have property δ if there exists $\delta > 0$ such that for every $x, y \in X$ with $\|x\| = \|y\| = 1$ and consecutive supports, that is $x = \sum_{i=1}^n a_i x_i$, $y = \sum_{i=n+1}^m b_i x_i$, $m \geq n - 1$, we have*

$$(2 - \delta)\|x + y\| \geq 2.$$

Brodskii and Milman [1] defined the concept of normal structure; the most relevant existence for non-expansive mappings is related to this notion restricted to weakly compact sets as was proved by Kirk [14], who showed that if T is a non-expansive mapping from a weakly compact convex set K with normal structure into itself, then it always has a fixed point.

Definition 2.4.

a) *A convex bounded subset K of a Banach space X has normal structure if and only if it does not contain a diametral sequence, that is if and only if there is no sequence $\{y_n\}_n$ such that*

$$\lim_n d(y_{n+1}, conv\{y_1, \ldots, y_n\}) = diam\{y_m\}_{m=1}^{\infty},$$

where $d(y, A)$ denotes the distance between y and the set A, conv A denotes the convex hull of A and diam A the diameter of A.

If every convex bounded subset of X has normal structure, we say that X has normal structure (NS).

b) *A space X has weak normal structure (WNS) if every weakly compact convex subset of X has normal structure, that is if there is no weakly null diametral sequence in X.*

We remark that all of the previously defined notions imply WNS as can be seen in [11], [12] and [7].

Another concept related to fixed point theory is the Kadec-Klee property:

Definition 2.5. *The norm in a Banach space X has the Kadec-Klee property (KK) if the weak topology and the norm topology coincide on the unit sphere of X.*

3. GGLD, δ, WNS et al.

In this section we study the relationship between the various properties defined above. First we start by giving an equivalent characterization of spaces with GGLD, which marks its similarity to spaces with WNS.

Domínguez Benavides et al. [5] proved that X has GGLD if and only if for every weakly null subsequence $\{y_n\}_n$,

$$\limsup_n \|y_n\| < A\{y_n\} = \lim_n \left(\sup_{i,j\geq n} \|y_i - y_j\|\right), \qquad (3.1)$$

where $A\{y_n\}$ is called the asymptotic diameter of $\{y_n\}_n$.

The referee pointed out to us that in [15] Sims and Smyth proved that GGLD is equivalent to a property called by them asymptotic *P*. This result in essence is the same as our proposition 3.4 (see also [16]). However our proof is not the same and is very simple.

Lemma 3.1. *Let $\{y_n\}_n$ be a weakly null sequence in a Banach space X. Then* $\limsup_n \|y_n\| \leq A\{y_n\}$.

Proof. If $A_n = \sup_{i,j\geq n} \|y_i - y_j\|$, $B_n = \sup\{\|y_j - y\| : y \in conv\{y_i\}_{i=n+1}^{\infty}, j \geq n\}$, clearly $A_n \leq B_n$ and $\lim_n A_n = A\{y_n\}$. On the other hand if $j \geq n$, $y = \sum_{i=n+1}^{r} \lambda_i y_i$, $\lambda_i \geq 0$ and $\sum_{i=n+1}^{r} \lambda_i = 1$, then

$$\left\| y_j - \sum_{i=n+1}^{r} \lambda_i y_i \right\| \leq \sum_{i=n+1}^{r} \lambda_i \|y_j - y_i\| \leq A_n.$$

Thus $Bn = A_n$ and since $y_n \xrightarrow{w} 0$, $0 \in \overline{conv}\{y_i\}_{i=n+1}^{\infty}$ for all n and $A_n = B_n \geq \|y_n\|$ for every n. Hence the result follows. □

This motivated us to define the following:

Definition 3.2. *A sequence $\{y_n\}_n$ in a Banach space X is called asymptotically diametral if* $\lim_n \|y_n\|$ *exists and*

$$\lim_n \|y_n\| = A\{y_n\},$$

where $A\{y_n\}$ is as in (3.1).

Corollary 3.3. *Let $\{y_n\}_n$ be a weakly null sequence in a Banach space X which is asymptotically diametral.*

a) *Then every subsequence of $\{y_n\}$ is asymptotically diametral and has the same asymptotic diameter as $\{y_n\}_n$.*

b) *If $\{\varepsilon_n\}_n \subset \mathbb{R}^+$ tends to zero and $\{w_n\}_n \subset X$ is such that $\|y_n - w_n\| < \varepsilon_n$, then $\{w_n\}_n$ is asymptotically diametral.*

Proof.

a) Let $\{y_{n_k}\}_k \subset \{y_n\}_n$ be a subsequence. Then

$$\lim_k \|y_{n_k}\| = \lim_k \|y_n\| = A\{y_n\} \geq A\{y_{k_k}\}$$

and by lemma 3.1,

$$\lim_k \|y_{n_k}\| = A\{y_{n_k}\}.$$

b) Since $\|y_n\| - \varepsilon_n \leq \|w_n\| \leq \|y_n\| + \varepsilon_n$ and $\|y_i - y_j\| - \varepsilon_i - \varepsilon_j \leq \|w_i - w_j\| \leq \|y_i - y_j\| + \varepsilon_i + \varepsilon_j$ we get that

$$\lim_n \|w_n\| = \lim_n \|y_n\| = A\{y_n\} = A\{w_n\}. \qquad \square$$

Proposition 3.4. *A Banach space X has GGLD if and only if it does not have weakly null asymptotically diametral sequences.*

Proof. If X does not have GGLD let $\{y_n\}_n$ be a weakly null sequence such that

$$\limsup_n \|y_n\| = A\{y_n\}.$$

Then there is a subsequence $\{y_{n_k}\}$ with $\lim_k \|y_{n_k}\| = A\{y_n\} \geq A\{y_{n_k}\}$ and by lemma 3.1

$$\lim_k \|y_{n_k}\| = A\{y_{n_k}\}.$$

The other implication is obvious. □

The following corollary is an immediate consequence of proposition 3.4 and corollary 3.3.

Corollary 3.5. *A Banach space X with a Schauder basis does not have GGLD if and only if there is a normalized asymptotically diametral weakly null block basis in X.*

Now we will start the study of the relationship between the different properties.

First observe that if X has a basis and GLD, then the basis is monotone since for every x, $\|P_n x\| \leq \|x\|$.

Proposition 3.6. *Let X be a Banach space with a Schauder basis. If X has GLD, then X has property δ. However there is a Banach space with a monotone Schauder basis, property δ and without GLD.*

Proof. Let $x, y \in X$ with consecutive support and $\|x\| = \|y\| = 1$. Let $\varepsilon = 1$, since X has GLD, there exists $r > 0$ such that $\|x + y\| \geq 1 + r$. Thus if $\delta < \dfrac{2r}{1+r}$

$$2 < (2 - \delta)(1 + r) \leq (2 - \delta)\|x + y\|.$$

Now let X be the completion of the space c_{00} of real sequences with finite support with the norm

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \max \left\{ |a_1|, \frac{|a_1|}{2} + |a_2| \right\} + \sum_{i=3}^{n} |a_i|.$$

Let $y = e_1 + \frac{1}{2} e_2$. Then $\|P_1 y\| = 1$, $\|(I - P_1)y\| = \frac{1}{2}$ and $\|y\| = 1$; thus X does not have GLD. However, if $x = \sum_{i=1}^{n} a_i e_i$, $y = \sum_{i=n+1}^{m} b_i e_i$, $\|x\| = \|y\| = 1$, then if $n > 1$, $2 = \|x + y\| = \|x\| + \|y\|$; if $x = a_1 e_1$ and $y = \sum_{i=2}^{m} b_i e_i$, then

$$\|x + y\| = \begin{cases} 1 + \sum_{i=3}^{m} |b_i| = 2 - |b_2| \geq \frac{3}{2} & \text{if } |b_2| \leq \frac{1}{2}. \\ \frac{|a_1|}{2} + \sum_{i=2}^{m} |b_i| = \frac{1}{2} + 1 = \frac{3}{2} & \text{if } |b_2| \geq \frac{1}{2}. \end{cases}$$

Hence property δ holds for $\delta = \frac{2}{3}$. □

Proposition 3.7. *Let X be a Banach space with Schauder basis. Then property δ implies GGLD but the inverse of this is false.*

Proof. Suppose $\{y_n\}_n \subset X$ is a weakly null block basis with $\|y_n\| = 1$. Since property δ holds, for $m \neq n$,

$$\|y_n - y_m\| \geq \frac{2}{2 - \delta}.$$

Thus

$$A\{y_n\} \geq \frac{2}{2 - \delta} > 1 = \lim_n \|y_n\|,$$

and by corollary 3.5, X has GGLD.

Now let $\{a_i\}_i \in c_{00}$ and let $\{a_i^*\}_i$ be the non-increasing rearrangement of the sequence $\{|a_i|\}_i$.

Define X to be the completion of c_{00} with the norm

$$\|\{a_i\}_i\| = \sum_{i=1}^{\infty} \frac{a_i^*}{i}.$$

Then it is shown in [8] that X does not have property δ and that X has WNS. The proof of this latter fact also shows that X has GGLD.

Jiménez-Melado in [12] proved the following:

Proposition 3.8. *Let X be a Banach space. If X has GGLD then X has WNS but the inverse of this is false.*

Now we turn our attention to the relations between KK versus GLD and GGLD.

Proposition 3.9. *Let X be a Banach space with a Schauder basis. Then, if X has GLD it also has KK but KK does not imply GGLD and thus in particular does not imply GLD.*

Proof. The proof that GLD implies KK is similar to that of *JT* having the Kadec-Klee property found in [6], observing that property GLD can be restated as: For every $\varepsilon > 0$ there exists $r > 0$ such that if $\|y\| = 1$ and for some $n \in \mathbb{N}$, $\|P_n y\| > 1 - r$, then $\|(I - P_n)y\| < \varepsilon$.

The counterexample for the second part is the following: Consider c_0 with the norm

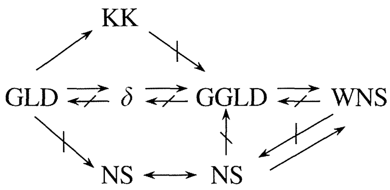
$$\|\{a_i\}_i\| = \left(\sum_{i=1}^{\infty} \frac{(a_i^*)^2}{2^i} \right)^{\frac{1}{2}},$$

where as before $\{a_i^*\}_i$, is the non-increasing rearrangement of $\{|a_i|\}_i$. This norm is equivalent to the usual norm in c_0 and thus does not have GGLD (see proposition 4.3) and in Diestel [4] it is shown to be locally uniformly convex and consequently has KK (see e.g. [2]).

Finally observe that l_1 is an example of a space with GLD and without NS. □

In the next section we complete the previous analysis by showing that there exists a space with NS, and thus with WNS, and without GGLD.

To wrap up this section schematically we present the following diagram:



4. GGLD, α' and c_0

One of the referees pointed out to us that in [9] García Falset et al. introduced property α', which is related to Kuratowski's measure of non compactness and implies WNS, and that they showed that neither c_0, nor l_1 can be renormed to have α'. Since it is evident that any Schur space has GGLD, GGLD does not imply α'. We do not know if α' implies GGLD, however we will see in this section that a stronger version of α', defined in [17], implies GGLD. Afterwards we will show that c_0 cannot be renormed in order to have GGLD.

Definition 4.1. *Let X be a Banach space.*

1. *Let $A \subset X$. The Kuratowski measure $\alpha(A)$ is defined as*

$$\alpha(A) = \inf \{r : A \textit{ can be covered by a finite number of sets of diameter} \leq r\}.$$

2. *The space X has property α' if there exists $0 < \delta < 1$ such that for every $f \in X^*$ with $\|f\| = 1$,*

$$\alpha\big(S(f, \delta)\big) < 1,$$

where $S(f, \delta) = \{x \in B_X : f(x) > 1 - \delta\}$.

3. *The space X has property strong a' if there exist $0 < \delta < 1$ and $0 < r < 1$ such that for every $f \in X^*$ with $\|f\| = 1$,*

$$\alpha\big(S(f, \delta)\big) < r.$$

The proof of the next theorem follows along the lines of the proof that α' implies WNS in [9].

Theorem 4.2. *Let X be a Banach space with property strong α'. Then X has GGLD.*

Proof. Suppose that X has *strong a'* and doesn't have GGLD. Then by corollary 3.3 there exists a weakly null asymptotically diametral sequence $\{x_n\}_n$ in X with $\|x_n\| = 1$, $n = 1, 2, \ldots$ and

$$1 = \lim_n \|x_n\| = \lim_n \sup_{i,j \geq n} \|x_i - x_j\| = A\{x_n\}.$$

We will see that $\alpha(\{x_n\}) = 1$. If this is not true, since clearly $\alpha(\{x_n\}) \leq A\{x_n\} = 1$, there exist $\varepsilon > 0$ and $B_1, \ldots, B_k$ with *diam* $B_i \leq 1 - \varepsilon$ so that $\{x_n\}_n \subset \cup_{i=1}^k B_i$. Thus there are i_0 and a subsequence $\{x_{n_j}\}_j$ so that $\{x_{n_j}\}_j$. Then

$$\lim_k \sup_{i,j \geq k} \|x_{n_i} - x_{n_j}\| \leq 1 - \varepsilon$$

and this contradicts the fact, proved in corollary 3.3, that every subsequence of an asymptotically diametral sequence is a also asymptotically diametral with the same asymptotic diameter as the original sequence. Hence $\alpha(\{x_n\}) = A\{x_n\} = 1$.

Since X has *strong* α' there exist $0 < \delta < 1$ and $0 < r < 1$ so that $\alpha\big(S(f, \delta)\big) < r$ for every $f \in X^*$ with $\|f\| = 1$. Let $f_0 \in X^*$ with $\|f_0\| = 1$ so that $f_0(x_1) = 1$. Let $\varepsilon > 0$ with $r < 1 - \varepsilon$ and $\varepsilon_1 = \min\left(\frac{\delta}{1-\delta}, \frac{\varepsilon}{1-\varepsilon}\right)$. Then, since $\{x_n - x_1\}_n$ converges weakly to $-x_1$,

$$-f_0\left(\frac{1}{1+\varepsilon_1}(x_n - x_1)\right) \xrightarrow[n \to \infty]{} \frac{1}{1+\varepsilon_1} > 1 - \delta.$$

Hence there exists n_1 so that

$$\left\{\frac{1}{1+\varepsilon_1} x_n : n \geq n_1\right\} \subset \frac{1}{1+\varepsilon_1} x_1 + S(-f_0, \delta)$$

and thus

$$\frac{1}{1+\varepsilon_1} = A\left\{\frac{1}{1+\varepsilon_1} x_n\right\} = \alpha\left(\left\{\frac{1}{1+\varepsilon_1} x_n\right\}\right) \leq$$

$$\leq \alpha\left(\frac{1}{1+\varepsilon_1} x_1 + S(-f_0, \delta)\right) = \alpha\big(S(-f_0, \delta)\big) < r < 1 - \varepsilon$$

which is a contradiction because $\frac{1}{1+\varepsilon_1} > 1 - \varepsilon$.

This proves that X has GGLD.

Proposition 4.3. *The space c_0 cannot be renormed in order to have GGLD.*

Proof. Let $\{e_i\}_i$ be the canonical basis in c_0 and suppose that the norm $\|\cdot\|$ in c_0 is such that

$$L \sup_i \|a_i\| \leq \left\|\sum_i a_i e_i\right\| \leq M \sup_i |a_i|.$$

Let $R_n = \sup\{\|\sum_{i=n}^r a_i e_i\| : \sup_i |a_i| = 1, r \geq n\}$. Clearly $R_n \geq R_{n+1}$ and $L \leq R_n \leq M$. Let $R = \lim_n R_n$. There exist r_1 and $\{a_i\}_{i=1}^{r_1}$ such that $\max_{1 \leq i \leq r_1} |a_i| = 1$ and

$$R_1 - 1 \leq \left\|\sum_{i=1}^{r_1} a_i e_i\right\| \leq R_1.$$

There exist $r_2 > r_1$ and $\{a_i\}_{i=r_1+1}^{r_2}$ with $\max_{r_1+1\le i\le r_2}|a_i| = 1$ and

$$R_{r_1+1} - \frac{1}{2} \le \left\| \sum_{i=r_1+1}^{r_2} a_i e_i \right\| \le R_{r_1+1}$$

and so forth, there exist $r_n > r_{n-1}$ and $\{a_i\}_{i=r_{n-1}+1}^{r_n}$ such that $\max_{r_{n-1}+1\le i\le r_n}|a_i| = 1$ and

$$R_{r_{n-1}+1} - \frac{1}{n} \le \left\| \sum_{i=r_{n-1}+1}^{r_n} a_i e_i \right\| \le R_{r_{n-1}+1}.$$

Let $u_n = \sum_{i=r_{n-1}+1}^{r_n} a_i e_i$, where $r_0 = 0$. Clearly $\|u_n\| \underset{n}{\to} R$, $u_n \overset{w}{\to} 0$ and if $i, j \ge n$

$$\|u_i - u_j\| \le R_{r_{n-1}+1}.$$

Hence $\lim_n(\sup_{i,j\ge n}\|u_i - u_j\|) \le R$. Thus by lemma 3.1, $\lim_n \|u_n\| = A\{u_n\} = R$ and by proposition 3.4 $(c_0, \|\cdot\|)$ does not have GGLD.

However Day, James and Swaminathan [3] proved that every separable space can be renormed to have NS and thus in particular to have WNS. Hence renorming C_0 properly we get a space with NS and without GGLD. □

Corollary 4.4. *Let X be a Banach space containing a subspace isomorphic to c_0. Then X does not have GGLD.*

5. GLD AND GGLD IN JAMES' SPACES

Based on Khamsi's paper [13] we proved in [6] that JT has WNS; in this section we study the relation of the generalized James and James tree spaces with regards to the GLD and GGLD properties.

We begin by remembering the definitions of those spaces.

Definition 5.1. *Let X be a Banach space with a normalized Schauder basis $\{x_i\}_i$.*

a) *$J(x_i)$ is the completion of c_{00} with the norm*

$$\left\| \sum_i a_i e_i \right\| = \sup \left\| \sum_{i=1}^{k} \left(\sum_{j=p_i}^{q_i} a_j \right) x_{p_i} \right\|_X,$$

where the sup is taken over all finite sequences of natural numbers with $1 \le p_1 \le q_1 < p_2 \le \cdots < p_k \le q_k$.

b) *The standard binary tree is $\mathcal{T} = \{(n, i) : 0 \le n < \infty, 0 \le i < 2^n\}$. The points (n, i) are called nodes; $(n + 1, 2i)$ and $(n + 1, 2i + 1)$ are the offspring of (n, i). A segment is a finite set $S = \{s_1, s_2, \ldots, s_m\}$ of nodes such that for every j, s_{j+1} is an offspring of s_j. We order $\mathcal{T} = \{t_i\}_{i=1}^{\infty}$ by $t_{2^i+j} = (i, j)$ for $i = 0, 1, \ldots, j = 0, \ldots, 2^i - 1$. Now we define two functions σ and μ from the set of segment to $\mathbb{N}$ as follows: If $S = \{t_{n_1}, t_{n_2}, \ldots, t_{n_m}\}$, where the subscripts of the nodes are those give by the order in $\mathcal{T}$, $\sigma(S) = n_1$ and $\mu(S) = n_m$. $JT(x_i)$ is the completion of c_{00} with the norm*

$$\left\| \sum_{t\in\mathcal{T}} a_t \eta_t \right\| = \sup \left\| \sum_{i=1}^{k} \left(\sum_{t\in S_i} a_t \right) x_{\sigma(S_i)} \right\|_X,$$

where the sup is taken over all collections $S_1, \ldots, S_k$ of finite disjoint segments in $\mathcal{T}$.

It is easy to see that $\{e_i\}_i$ is a monotone Schauder basis in $J(x_i)$ and $\{\eta_t\}_{t\in\mathcal{T}}$ is a monotone Schauder basis in $JT(x_i)$.

Proposition 5.2. *If X has a normalized Schauder basis $\{x_i\}_i$ and GGLD (GLD), then $JT(x_i)$ also has GGLD (GLD).*

Proof. Let $\{u_n\}_n$ be a normalized weakly null block basic sequence in $JT(x_i)$ with $u_n = \sum_{i=p_n}^{q_n} a_{t_i}\eta_{t_i}$. Then there exist $\{k_i\}_i$, and disjoint segments S_i^n such that $p_n \le \sigma(S_i^n) \le \le \mu(S_i^n) \le q_n$ for $i = 1, \ldots, k_n$ and

$$1 = \|u_n\| = \left\| \sum_{i=1}^{k_n} \left(\sum_{t\in S_i^n} a_t \right) x_{\sigma(S_i^n)} \right\|_X.$$

Now let $v_n = \sum_{i=1}^{k_n} (\sum_{t\in S_i^n} a_t) x_{\sigma(S_i^n)} \in X$. We will see that $\{v_n\}_n$ is weakly null in X. Define $U : [u_n] \to X$ by $U(u_n) = = v_n$ where $[u_n]$ is the closed linear span of $\{u_n\}_n$. Then by the definition of the norm

$$\left\| U\left(\sum_n b_n u_n \right) \right\|_X = \left\| \sum_n b_n v_n \right\|_X =$$

$$= \left\| \sum_n \sum_{i=1}^{k_n} \left(\sum_{t\in S_i^n} b_n a_t \right) x_{\sigma(S_i^n)} \right\|_X \le \left\| \sum_n b_n u_n \right\|.$$

Hence U is continuous and $\{v_n\}_n$ *is weakly null in X.*

In particular, for all n, m

$$\|u_n - u_m\| \ge \|v_n - v_m\|_X.$$

Since X has GGLD, $A\{u_n\} \ge A\{v_n\} > 1 = \lim_n \|v_n\|_X = = \lim_n \|u_n\|$ and by corollary 3.5, $JT(x_i)$ has GGLD.

Now suppose X has GLD and let $\varepsilon > 0$ and $u_1, u_2 \in JT(x_i)$, with consecutive supports and $\|u_1\| = 1$, $\|u_2\| > \varepsilon$. Suppose $u_1 = \sum_{i=1}^{n} a_{t_i}\eta_{t_i}$, $u_2 = \sum_{i=n+1}^{m} a_{t_i}\eta_{t_i}$. As before, there exist v_1 and v_2 in X with consecutive supports and $\|u_1\| = = \|v_1\|_X$ and $\|u_2\| = \|v_2\|_X$. Since X has GLD there exists $r > 0$ so that

$$1 + r \le \|v_1 + v_2\|_X \le \|u_1 + u_2\|$$

and this finishes the proof. □

Since $J(x_i)$ is isometrically isomorphic to a subspace of $JT(x_i)$, via an isomorphism U such that $\{Ue_i\}_i$ is a subsequence of $\{\eta_t\}_{t\in\mathcal{T}}$, we get the following corollary:

Corollary 5.3. *If X has a normalized Schauder basis $\{x_i\}_i$ and GGLD (GLD), then $J(x_i)$ also has GGLD (GLD).*

Remark 1. *There are other generalizations of J:*

a) *The space $J_1(x_i)$ is the completion of c_{00} with the norm*

$$\left\|\sum_i a_i e_i\right\|_1 = \sup \left\|\sum_{i=1}^{k} (a_{p_{2i}} - a_{p_{2i-1}})x_{p_{2i-1}}\right\|_X,$$

where the sup is taken over all finite sequences of natural numbers with $1 \leq p_1 < p_2 \leq \cdots < p_{2k}$.

b) *The space $J_2(x_i)$ is the completion of c_{00} with the norm*

$$\left\|\sum_i a_i e_i\right\|_2 = \sup \left\|\sum_{i=1}^{k} (a_{p_{i+1}} - a_{p_i})x_{p_i}\right\|_X,$$

where the sup is taken over all finite sequences of natural numbers with $1 \leq p_1 < p_2 \leq \cdots < p_{k+1}$.

Modifying the definition of GLD slightly as follows: if for each $\varepsilon > 0$, there exists $r > 0$ such that for every $x \in X$ and $n \in \mathbb{N}$ we have

$$\|P_n x\| = 1 \text{ and } \|(I - P_{n+1})x\| \geq \varepsilon \text{ implies } \|x\| \geq 1 + r,$$

then proposition 3.6 remains true.

Using this we can prove with an argument similar to that employed in the proof of proposition 5.2, that if X has GGLD (GLD), then $J_1(x_i)$ and $J_2(x_i)$ have GGLD (modified GLD). In particular taking X as l_2, it follows that James space J has modified GLD; although it does not have GLD.

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