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RISK AND UNCERTAINTY AVERSION WITH MULTIDIMENSIONAL CONSEQUENCES

(Risk/uncertainty/aversion/multidimensional consequences)

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1. INTRODUCTION

In a preceding paper (Montesano, 1999b) a systematic set of definitions on risk and uncertainty aversion was introduced with regard to unidimensional lotteries and acts. Taking into account only the preference preordering over the set of all possible lotteries and acts represented by the certainty equivalent function, many propositions were introduced and demostrated on global and local risk and uncertainty aversion, comparative risk and uncertainty aversion, and aversion to increasing risk and uncertainty.

In this paper the preceding analysis is extended to the multidimensional case, i.e., to lotteries and acts whose consequences are points of a k-dimensional Euclidean space, which can be interpreted as bundles with k commodities. The analysis takes into consideration finite lotteries and acts, i.e., the number of the possible states of nature is finite. No specific model for the decision-maker preferences is assumed, but only the existence of a regular preference system represented by an ordinal utility function. Subsequently some specific models are introduced (the Expected Utility model, the Rank Dependent Expected Utility model, and the Choquet Expected Utility model), in order to extend some results already examined (Montesano, 1999a) for the unidimensional case.

In Section 2 the Decision Making Under Risk (DMUR) and Uncertainty (DMUU) situations are introduced, together with the main assumptions and some definitions. In particular, mean preserving spreads are also introduced for the multidimensional case, while their current definition only applies to the unidimensional case. Section 3 examines the DMUR situation. Aversion to risk and to increasing risk are introduced and some propositions are demonstrated which relate these aversions to some characteristics of the preference system. Unlike the unidimensional case, the notion of comparative risk aversion (as well as that of comparative uncertainty aversion) is unattractive, since it requires that the preferences of two decision-makers coincide over the degenerate lotteries set. This is an overly compelling assumption, which discourages further analysis. Section 4 concerns the DMUU situation, where both risk and uncertainty matter. Aversion to them is distinctly introduced and some propositions on these aversions are proposed.

Although not all the propositions found for the unidimensional case can be extended to the multidimensional case, many of them surprisingly can, providing us with tools for investigating real decision-makers' characteristics.

2. SOME INTRODUCTORY DEFINITIONS AND ASSUMPTIONS

A Decision Making Under Uncertainty (DMUU) situation is represented by a quadruple $\langle S; \Omega; F; X\rangle$ where S is a set of states of nature, Ω is an algebra on S, X is a set of consequences and F is the set of possible acts (i.e., functions from S to X). In this paper S is finite, $\Omega = 2^S$ (the empty set included), and X is a compact and convex subset of the Euclidean k-dimensional space $\mathbb{R}^k$.

A Decision Making Under Risk (DMUR) situation is represented by a quintuple $\langle S; \Omega; F; X; p\rangle$, where p is an exogenous probability: $p \in P$, where P is the set of all possible probability distributions over $S = \{s_1, ..., s_m\}$, i.e., $P = \{p(s_j) \geqslant 0$ for $j = 1, ..., m : \Sigma_{j=1}^m p(s_j) = 1\}$ or $P = \{p(e_i) \geqslant 0$ for every $e_i \in 2^S : p(\emptyset) = 0$, $p(S) = 1$, and $p(e_i \cap e_g) + p(e_i \cup e_g) = p(e_i) + p(e_g)$ for every pair $e_i, e_g \in 2^S\}$, so that $\Sigma_{i=1}^n p(e_i) = 1$ if $(e_i)_{i=1}^n$ is a partition of S. The set Q of lotteries on X is the set of probability measures on X induced through the acts by some probability, i.e., $q = (f, p)$, where $f \in F$ and $p \in P$, and $Q = F \times P$.

Consequently, and act $f \in F$ can be represented by $f = (x(s_1), ..., x(s_m))$ or $f = (x(s_j))_{j=1}^m$, where m is the number of the possible states of nature; a lottery $q \in Q$ can be represented by $q = (x(s_j), p(s_j))_{j=1}^m$, where $p(s_j) \geqslant 0$ is the probability of the state of nature s_j, with $\Sigma_{j=1}^m p(s_j) = 1$. Let $e : X \to S$ be the inverse correspondence of fuction $f : S \to X$, i.e., $e(x_i) = \{s_j \in S : x(s_j) = x_i\}$. Therefore, an act f can also be represented by $f = (x_i, e(x_i))_{i=1}^n$, where $(e(x_i))_{i=1}^n$ is the event associated with the possible consequence $x_i \in X$ and n is the number, with $n \leqslant m$, of the possible consequences of the act, thus a variable depending on the act under consideration. Analogously, a lottery can also be represented by $q = (x_i, p(e(x_i)))_{i=1}^n$ where $p(e(x_i)) = \Sigma_{s_j \in e(x_i)} p(s_j)$ is the probability of the event e_i or by $q = (x_i, p(e(x_i)))_{i=1}^n$ or $q = (x_i, p_i)_{i=1}^n$, where $p_i = p(e(x_i))$ is the probability of the possible consequence $x_i \in X$, with $\Sigma_{i=1}^n p_i = 1$. Let F_n and Q_n respectively indicate the set of all acts and the set of all lotteries with n different possible consequences: thus, $F = \bigcup_{n=1}^m F_n$ and $Q = \bigcup_{n=1}^m Q_n$.

A decision-maker in DMUU situations is represented by the preference system $\langle F, \gtrsim \rangle$ where $\gtrsim$ is a preference preordering over F. A decision-maker in DMUR situations is represented by the preference system $\langle Q, \gtrsim \rangle$.

Assumption 1. The preference system $\langle F, \gtrsim \rangle$ can be represented by means of an (ordinal) bounded utility function $u : F \to \gtrsim$, i.e., $f \leqslant f'$ if and only if $u(f) \geqslant u(f')$. Let us indicate with $u(x)$ the utility of the degenerate act $f = (x, S)$, where $x \in X$. Analogously the preference system $\langle Q, \gtrsim \rangle$ can be represented by means of an (ordinal) bounded utility function $u : Q \to \mathbb{R}$, i.e, $q \gtrsim q'$ if and only if $u(q) \geqslant u(q')$ and $u(x)$ is the utility of the degenerate lottery $q = (x, 1)$.

Assumption 2. The ordinal utility functions $u : F_n \to \mathbb{R}$ and $u : Q_n \to \mathbb{R}$ are differentiable and monotonically weakly increasing functions of $(x_i)_{i=1}^n$ for every $n = 1, ..., m$ (where «monotonically weakly increasing» here means that the gradient of u with respect to $(x_i^h)_{i=1}^n$ is a semipositive vector for every $h = 1, ..., k$).

Remark: The assumption that $u(f)$ and $u(q)$ are differentiable with respect to $(x_i)_{i=1}^n$ for every $f \in F_n$ and $q \in Q_n$ does not imply that they are differentiable with respect to $(x_i)_{i=1}^n \in X^n$ at the points of X^n where two, or more than two, possible consequences coincide (if this happens, then $f \in F_{n'}$ and $q \in F_{n'}$ with $n' < n$).

While Assumption 1 is required to hold throughout the paper, Assumption 2 is required only by some propositions and it is specifically recalled when adopted.

Definition 1. (Expected consequence): The expected consequence function $E : Q \to X$, or, equivalently, $E : F \times P \to X$ defines $E(q) = \Sigma_{i=1}^n p_i x_i$, where $q = (x_i, p_i)_{i=1}^n$, or $E(f, p) = \Sigma_{i=1}^n p(e_i) x_i$ where $f = (x_i, e_i)_{i=1}^n$ and p is a probability distribution. Note that the expected consequence function is onto.

Definition 2. (Value of a commodity in an act or a lottery): If Assumption 2 holds, we can introduce, for every commodity $h = 1, ..., k$ with respect to an act $f \in F$ or a lottery $q \in Q$, a normalized marginal value $r^h(f)$ or $r^h(q)$ through the relationships

$$r^h(f) = \frac{\sum_{i=1}^n \frac{\partial u(f)}{\partial x_i^h}}{\sum_{s=1}^k \sum_{v=1}^n \frac{\partial u(f)}{\partial x_v^s}} \qquad r^h(q) = \frac{\sum_{i=1}^n \frac{\partial u(q)}{\partial x_i^h}}{\sum_{s=1}^k \sum_{v=1}^n \frac{\partial u(q)}{\partial x_v^s}}$$

Assumption 2 implies $r^h(f) > 0$ and $r^h(q) > 0$ for every $h = 1, ..., k$, while normalization means $\Sigma_{h=1}^k r^h(f) = 1$ and $\Sigma_{h=1}^k r^h(q) = 1$. If we take under consideration the acts $f(t)$ or lotteries $q(t)$, where $t \in [0, 1]$, $f(t) = (x_i(t), e_i)_{i=1}^n$ and $q(t) = (x_i(t), p_i)_{i=1}^n$ with $x_i(t) = x + t(x_i - x)$, we can introduce the local value $r^h(x) = \lim_{t\to 0} r^h(f(t)) = \lim_{t\to 0} r^h(q(t))$: we find

$$r^h(x) = \frac{\frac{\partial u(x)}{\partial x^h}}{\sum_{s=1}^k \frac{\partial u(x)}{\partial x^s}}$$

since $u(x) = \lim_{t\to 0} u(f(t)) = \lim_{t\to 0} u(q(t))$, so that

$$\frac{\partial u(x)}{\partial x^h} = \lim_{t\to 0} \sum_{i=1}^n \frac{\partial u(f(t))}{\partial x_i^h(t)} (1 - t) = \lim_{t\to 0} \sum_{i=1}^n \frac{\partial u(f(t))}{\partial x_i^h(t)}$$

and, analogously,

$$\frac{\partial u(x)}{\partial x^h} = \lim_{t\to 0} \sum_{i=1}^n \frac{\partial u(q(t))}{\partial x_i^h(t)}.$$

Note that local values of commodities only depend on the limit consequence x.

Definition 3. (Expected value): The expected value function $EV : Q \to \mathbb{R}$ or, equivalently, $EV : F \times P \to \mathbb{R}$ defines $EV(q) = r(q)E(q)$, i.e., $EV(q) = \Sigma_{h=1}^k \Sigma_{i=1}^n r^h(q) p_i x_i^h$, or $EV(f, p) = r(f)E(f, p)$. We can also introduce the expected local value functions $EV(x, q) = r(x)E(q)$ and $EV(x, f, p) = r(x)E(f, p)$.

Definition 4. (Marginal weight of an event for a commodity in an act or lottery): If Assumption 2 holds, we can introduce for every event $i = 1, ..., n$ and for every commodity $h = 1, ..., k$ with respect to an act $f \in F$ or a lottery $q \in Q$, a normalized marginal weight $p_i^h(f)$ or $p_i^h(q)$ through the relationships

$$p_i^h(f) = \frac{\frac{\partial u(f)}{\partial x_i^h}}{\sum_{v=1}^n \frac{\partial u(f)}{\partial x_v^h}} \qquad p_i^h(q) = \frac{\frac{\partial u(q)}{\partial x_i^h}}{\sum_{v=1}^n \frac{\partial u(q)}{\partial x_v^h}}$$

Assumption 2 implies $p_i^h(f) \geqslant 0$ and $p_i^h(q) \geqslant 0$ for every $i = 1, ..., n$ and $h = 1, ..., k$, while normalization means that $\Sigma_{i=1}^n p_i^h(f) = 1$ and $\Sigma_{i=1}^n p_i^h(q) = 1$. If we take into consideration the acts $f(t) = (x_i(t), e_i)_{i=1}^n$ and the lotteries $q(t) = (x_i(t), p_i)_{i=1}^n$, with $x_i(t) = x + t(x_i - x)$, we can introduce the local weight $p_i^h(x, f) = \lim_{t\to 0} p_i^h(f(t))$ and $p_i^h(x, q) = \lim_{t\to 0} p_i^h(q(t))$, with

$$p_i^h(x,f) = \frac{1}{\frac{\partial u(x)}{\partial x^h}} \lim_{t\to 0} \frac{\partial u(f(t))}{\partial x_i^h(t)} \quad p_i^h(x, q) = \frac{1}{\frac{\partial u(x)}{\partial x^h}} \lim_{t\to 0} \frac{\partial u(q(t))}{\partial x_i^h(t)}.$$

Note that local weights of events generally depend not only on the limit consequence x, but also on the acts $f = (x_i, e_i)_{i=1}^n$ or lotteries $q = (x_i, p_i)_{i=1}^n$.

Definition 5. (Marginal Expected consequence): Introducing marginal weights of events in place of probabilities we have the marginal expected consequence functions $ME : F \to X$ and $ME : Q \to X$, with $ME(f) = (\Sigma_{i=1}^n p_i^h(f)x_i^h)_{h=1}^k$ and $ME(q) = (\Sigma_{i=1}^n p_i^h(q)x_i^h)_{h=1}^k$. We can also introduce the local marginal expected consequence functions $ME(x, f) = (\Sigma_{i=1}^n p_i^h(x, f)x_i^h)_{h=1}^k$ and $ME(x, q) = (\Sigma_{i=1}^n p_i^h(x, q)x_i^h)_{h=1}^k$.

Definition 6. (Marginal Expected Value): The marginal expected value function $MEV : F \to \mathbb{R}$ or $MEV : Q \to \mathbb{R}$ defines $MEV(f) = r(f)ME(f)$, i.e, $MEV(f) = \Sigma_{h=1}^k \Sigma_{i=1}^n r^h(f)p_i^h(f)x_i^h$, or $MEV(q) = r(q)ME(q) = \Sigma_{h=1}^k \Sigma_{i=1}^n r^h(q)p_i^h(q)x_i^h$. We can also introduce the local marginal expected value functions, $MEV(x, f) = r(x)ME(x, f)$ and $MEV(x, q) = r(x)ME(x, q)$.

Definition 7. (Probabilistic mixture of two acts or two lotteries): for $f_a = (x_a(s_j))_{j=1}^m, f_b = (x_b(s_j))_{j=1}^m$ and $\lambda \in [0, 1]$, we indicate with $\lambda f_a \oplus (1 - \lambda)f_b$ the act $(q(s_j))_{j=1}^m$, where $q(s_j) = (x_a(s_j), \lambda; x_b(s_j), 1 - \lambda)$ is the lottery that, if s_j occurs, gives $x_a(s_j)$ or $x_b(s_j)$ with probabilities λ, $1 - \lambda$. The set F of all possible acts is assumed to include also the probabilistic mixture of acts and the preference system $\langle F, \gtrsim \rangle$ can be represented by an ordinal utility function even for these mixtures. Analogously, for $q_a = (x_a(s_j), p(s_j))_{j=1}^m$, $q_b = (x_b(s_j), p(s_j))_{j=1}^m$, and $\lambda \in [0, 1]$, we indicate with $\lambda q_a \oplus (1 - 1)q_b$ the lottery $(q(s_j), p(s_j))_{j=1}^m$, where $q(s_j) = (x_a(s_j), \lambda; x_b(s_j), 1 - \lambda)$, i.e. $\lambda q_a \oplus (1 - \lambda)q_b = (\lambda f_a \oplus (1 - \lambda)f_b, p)$.

Definition 8. (Set of the mean-preserving-anti-spreads lotteries with respect to the lottery q^*): $MPAS(q^*) = \{q \in Q : E(q) = E(q^*)$ and $p(x_i) = p^*(x_i)$ for all i except three points $x_a, x_b, x_c \in X$, with $x_b = \lambda x_a + (1 - \lambda)x_c$, $\lambda \in (0, 1)$ and $p(x_b) \geqslant p^*(x_b)\}$. This definition, which says that $q \in MPAS(q^*)$ is less riskier than q^*, is an extension to the multivariate case of the current definition of mean-preserving-spreads applied to the univariate case. However, we introduce a variation of probabilities which increases the probability of an intermediate consequence in place of a variation which decreases it. We do this since we can have $p^*(x_b) = 0$ for $x_b = \lambda x_a + (1 - \lambda)x_c$ and $\lambda \in (0, 1)$. Note that if q^* has n consequences to which a positive probability is associated, $q \in MPAS(q^*)$ can have $n - 2$, $n - 1$, n, or $n + 1$ consequences. Note also that this definition is independent from decision-maker's preferences.

Definition 9. (Set of the lotteries with a utility at most, or at least, as high as $u \in \mathbb{R}$):

$$G_Q(u) = \{q \in Q : u(q) \leqslant u\}$$
$$\bar{G}_Q^C(u) = \{q \in Q : u(q) \geqslant u\}$$

Definition 10. (Set of the lotteries with a utility of the expected consequence at most, or at least, as high as $u \in \mathbb{R}$):

$$H_Q(u) = \{q \in Q : u(E(q)) \leqslant u\}$$
$$\bar{H}_Q^C(u) = \{q \in Q : u(E(q)) \geqslant u\}$$

where $u(E(q))$ is the utility of the degenerate lottery $(E(q), 1)$.

Definition 11. (Set of the acts with a utility at most, or at least, as high as $u \in \mathbb{R}$):

$$G_F(u) = \{f \in F : u(f) \leqslant u\}$$
$$\bar{G}_F^C(u) = \{f \in F : u(f) \geqslant u\}$$

Definition 12. (Set of the acts with a utility of the expected consequence for probability p at most as high as $u \in \mathbb{R}$):

$$H_F(u, p) = \{f \in F : u(E(f, p)) \leqslant u\}$$

Definition 13. (Set of the acts through which the probability p induces lotteries with a utility at most as high as $u \in \mathbb{R}$):

$$L_F(u, p) = \{f \in F : u(f, p) \leqslant u\}$$

3. RISK AVERSION IN A DECISION MAKING UNDER RISK SITUATION

3.1. Definitions of global risk aversion and aversion to increasing risk

Definition 14. (Global risk aversion): $\langle Q, \gtrsim \rangle$ exhibits risk aversion if $(E(q), 1) \gtrsim q$ (i.e., if $u(E(q)) \geqslant u(q)$ for all $q \in Q$); risk attraction if $(E(q), 1) \lesssim q$; and risk neutrality if $(E(q), 1) \sim q$, where $E(q)$ is the Expected consequence of q introduced by Definition 1.

Definition 15. (Attraction to MPAS (mean-preserving-anti-spreads)-decreasing risk): $\langle Q, \gtrsim \rangle$ exhibits attraction to MPAS-decreasing risk if $q \lesssim q^*$ for all $q \in MPAS(q^*)$; aversion if $q \lesssim q^*$; and neutrality if $q \sim q^*$, where $MPAS(q^*)$ is the set of mean-preserving-anti-spreads lotteries with respect to q^* introduced by Definition 8.

Remark: Attraction to MPAS-decreasing risk implies risk aversion. We can see it by considering that the degenerate lottery $\big(E(q), 1\big)$ can be generated through a sequence of mean-preserving-anti-spreads lotteries starting from any lottery $q = (x_i, p_i)_{i=1}^n$: let us take under consideration the sequence $q_2, ..., q_n$ where $q_t = (x'_t, \Sigma_{v=1}^t p_v; x_{t+1}, p_{t+1}; ...; x_n, p_n)$ and $x'_t = \frac{1}{\Sigma_{v=1}^t p_v}(x'_{t-1} \Sigma_{v=1}^{t-1} p_v + x_t p_t)$ with $x'_1 = x_1$. We find that $E(q_t) = E(q)$ and $q_t \in MPAS(q_{t-1})$ for every $t = 2, ..., n$, with $q_n = \big(E(q), 1\big)$. Consequently, since attraction to MPAS-decreasing risk requires $q_t \succsim q_{t-1}$ for every $t = 2, ..., n$, then $q_n \succsim q_1$, i.e. $\big(E(q), 1\big) \succsim q_1 = q$.

Definition 16. (Aversion to PM (probabilistic mixture)-increasing risk): $\langle Q, \succsim \rangle$ exhibits aversion to PM-increasing risk if $u(\lambda q_a \oplus (1-\lambda)q_b) \leqslant \max\{u(q_a), u(q_b)\}$ for all $q_a, q_b \in Q$ and $\lambda \in [0, 1]$; attraction if $u(\lambda q_a \oplus (1-\lambda)q_a) \geqslant \min\{u(q_a), u(q_b)\}$; and neutrality if both aversion and attraction.

3.2. Restrictions of comparative risk aversion analysis in the multivariate case

Definition 17. (Comparative risk aversion): The natural definition of comparative risk aversion states that decision-maker $\langle Q, \succsim_A \rangle$ is more risk averse than decision-maker $\langle Q, \succsim_B \rangle$ if $(x, 1) \succsim_B q$ implies $(x, 1) \succsim_A q$ and $(x, 1) \precsim_A q$ implies $(x, 1) \precsim_B q$ for all $q \in Q$.

Proposition 1. Definition 17 implies that both decision-maker's preferences are equally ordered on the degenerate lotteries set, i.e., $u_A(x') \geqslant u_A(x)$ if and only if $u_B(x') \geqslant u_B(x)$ for all pairs $x, x' \in X$.

Proof: Let us introduce the sets:

$$R(q) = \{x \in X : (x, 1) \succsim q\}$$
$$\bar{R}^C(q) = \{x \in X : (x, 1) \precsim q\}$$

Definition 17 say that $\langle Q, \succsim_A \rangle$ is more risk averse than $\langle Q, \succsim_B \rangle$ if $R_B(q) \subseteq R_A(q)$ and $\bar{R}_A^C(q) \subseteq \bar{R}_B^C(q)$ for all $q \in Q$. Proposition 1 states that Definition 17 implies $R_A(x) = R_B(x)$ for all $x \in X$. Suppose not. Then, there is a $x' \in X$ with $R_B(x') \subseteq R_A(x')$ and $R_A(x') \neq R_B(x')$, i.e., there is a $\hat{x} \in X$ with $\hat{x} \in R_A(x')$ and $\hat{x} \notin R_B(x')$, i.e., $\hat{x} \succsim_A x'$ and $\hat{x} \prec_B x'$, so that $x' \in R_B(\hat{x}) \subseteq R_A(\hat{x})$, $x' \notin \bar{R}_B^C(\hat{x})$ and $x' \notin R_B(x')$, i.e., $\hat{x} \succsim_A x'$ and $\hat{x} \prec_B x' \in R_B(\hat{x}) \subseteq R_A(\hat{x})$, $x' \notin \bar{R}_B^C(\hat{x})$ and $x' \in \bar{R}_A^C(\hat{x})$, while Definition 17 requires $\bar{R}_A^C(\hat{x}) \subseteq \bar{R}_B^C(\hat{x})$, i.e., that $x' \precsim_A \hat{x}$ implies $x' \precsim_B \hat{x}$. Therefore, if there is a $x' \in X$ with $R_A(x') \neq R_B(x')$, then the condition $R_B(q) \subseteq R_A(q)$ and $\bar{R}_A^C(q) \subseteq \bar{R}_B^C(q)$ for all $q \in Q$ is impossible. □

This result coincides with the classical Kihlstrom and Mirman (1974) indication. Proposition 1 makes the comparative risk aversion of Definition 17 so restrictive that it is better to waive it, at least in the present paper.

3.3. Two propositions on global risk aversion

Proposition 2. $\langle Q, \succsim \rangle$ exhibits risk aversion (introduced by Definition 14) if and only if $H_Q(u) \subseteq G_Q(u)$ for all $u \in \mathbb{R}$; attraction if and only if $H_Q(u) \supseteq G_Q(u)$; and neutrality if and only if $H_Q(u) = G_Q(u)$.

Proof: Let us first demostrate the necessary condition for risk aversion. If the condition $H_Q(u) \subseteq G_Q(u)$ is not satisfied for all $u \in \mathbb{R}$, then there are a $u^* \in \mathbb{R}$ and a $q^* \in H_Q(u^*)$ such that $q^* \notin G_Q(u^*)$, i.e., $u(q^*) > u^*$ while $u\big(E(q^*)\big) \leqslant u^*$, so that $u\big(E(q^*)\big) < u(q^*)$, i.e., $E(q^*) \prec q^*$. Let us now demonstrate the sufficient condition. Since $H_Q(u) \subseteq G_Q(u)$ for all $u \in \mathbb{R}$ and $q \in H_Q\big(u(E(q))\big)$ for all $q \in Q$, then it is also $q \in G_Q\big(u(E(q))\big)$, i.e., $u(q) \leqslant u\big(E(q)\big)$ for all $q \in Q$. Analogously for risk attraction. The condition for risk neutrality is deduced taking into account that neutrality means that there is both aversion and attraction. □

Graphic representation: With reference to the Marschak-Machina diagram, where $n = 3$ and consequences are given, i.e., $q = (x_1, p_1; x_2, p_2; x_3, p_3)$ with $p_1 + p_2 + p_3 = 1$, if $x_1 \succ x_2 \succ x_3$ and $x_1 \succsim q^* \succsim x_3$ for every $q^* = (x_i, p_i)_{i=1}^3$, then there is global risk aversion if the indifference curve $q \sim q^*$ is southeast with respect to the indifference curve $E(q) \sim q^*$ (as represented in Figure 1). We have this since both the indifference curves $q \sim q^*$ and $E(q) \sim q^*$ divide the triangle into two regions, respectively $G_Q(u(q^*))$, $\bar{G}_Q^C\big(u(q^*)\big)$ and $H_Q\big(u(q^*)\big)$, $\bar{H}_Q^C\big(u(q^*)\big)$ with point $(0, 1) \in G_Q\big(u(q^*)\big)$ and $(1,0) \in \bar{G}_Q^C\big(u(q^*)\big)$ (note that points (0, 1) and (1, 0) respectively represent the degenerate lotteries $(x_3, 1)$ and $(x_1, 1)$). Consequently $H_Q\big(u(q^*)\big) \subseteq G_Q\big(u(q^*)\big)$ implies that the indifference curve $q \sim q^*$ is below the curve $E(q) \sim q^*$.

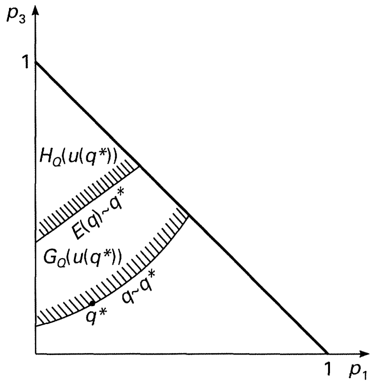

Figure 1

The Hirshleifer-Yaari diagram (where $n = 2$ and probabilities are given) can be drawn only if $k = 1$, i.e., only in the univariate case.

Proposition 3. If Assumption 2 holds, then $\langle Q, \succsim \rangle$ exhibits risk aversion if $EV(q) \geqslant MEV(q)$ for all $q \in Q_n$ and $n = 1, ..., m$ (where the Expected Value function $EV(q)$ is introduced by Definition 3 and the Marginal Expected Value $MEV(q)$ is introduced by Definition 6); risk attraction if $EV(q) \leqslant MEV(q)$; and risk neutrality if $EV(q) = MEV(q)$.

Proof: Let us intoduce for every $q \in Q_n$ and $t \in (0, 1]$ the lottery $q(t) = \left(x_i(t), p_i\right)_{i=1}^n$, where $x_i(t) = tx_i + (1 - t)E(q)$, and the utility risk premium function $RP_u(t; q) = u\big(E(q(t))\big) - u\big(q(t)\big)$. We find that $E\big(q(t)\big) = E(q)$ for all $t \in [0, 1]$ and, taking into account Definitions 2, 3, 4, 5 and 6, $\frac{dRP_u(t;\ q)}{dt} = -\frac{du\big(q(t)\big)}{dt} = -\Sigma_{h=1}^k \Sigma_{i=1}^n \frac{\partial u\big(q(t)\big)}{\partial x_i^h(t)} \big(x_i^h - E^h(q)\big) = -\Sigma_{h=1}^k\ r^h\big(q(t)\big)\ \Sigma_{i=1}^n\ p_i^h\big(q(t)\big)\big(x_i^h - E^h(q)\big)\ \Sigma_{s=1}^k\ \Sigma_{v=1}^n \frac{\partial u\big(q(t)\big)}{\partial x_v^s(t)} = \frac{1}{t}\left(EV\big(q(t)\big) - MEV\big(q(t)\big)\right) \Sigma_{s=1}^k\ \Sigma_{v=1}^n \frac{\partial u\big(q(t)\big)}{\partial x_v^s(t)}$, since $ME^h\big(q(t)\big) = \Sigma_{i=1}^n\ p_i^h\big(q(t)\big)x_i^h(t) = t\Sigma_{i=1}^n\ p_i^h\big(q(t)\big)x_i^h + (1 - t)E^h(q)$ so that $\frac{1}{t}\left(E^h(q) - ME^h\big(q(t)\big)\right) = E^h(q) - \Sigma_{i=1}^n p_i^h\big(q(t)\big)x_i^h = -\ \Sigma_{i=1}^n\ p_i^h\big(q(t)\big)\big(x_i^h - E^h\ (q)\big)$ and $\frac{1}{t}\left(EV\big(q(t)\big) - MEV\big(q(t)\big)\right) = \frac{1}{t}\ \Sigma_{h=1}^k\ r^h\big(q(t)\big)\left(E^h(q) - ME^h\big(q(t)\big)\right) = -\Sigma_{h=1}^k r^h\big(q(t)\big)\ \Sigma_{i=1}^n\ p_i^h\big(q(t)\big)\big(x_i^h - E^h(q)\big)$. Consequently, if $EV(q) \geqslant MEV(q)$ for all $q \in Q_n$, then $EV\big(q(t)\big) \geqslant MEV\big(q(t)\big)$ for all $t \in (0, 1]$, so that $\frac{dRP_u(t;\ q)}{dt} \geqslant 0$ for all $t \in (0, 1]$. Thus, since $\lim_{t \to 0} RP_u(t;\ q) = 0$ and $\frac{dRP_u(t;\ q)}{dt} \geqslant 0$ for $t \in (0, 1]$, then $RP_u(1;\ q) \geqslant 0$, i.e., $u\big(E(q)\big) \geqslant u(q)$ for all $q \in Q_n$ and $n = 1, ..., m$. □

3.4. Local risk aversion and two other propositions on global risk aversion

Definition 18. (Local risk aversion): $\langle Q, \succsim \rangle$ exhibits local risk aversion if for every $x \in X$ and $q \in Q$ there is a $t^* > 0$ such that $u\big(E(q(t))\big) \geqslant u\big(q(t)\big)$ for every $t \in [0, t^*]$, where $q(t) = \big(x_i(t), p_i\big)_{i=1}^n$ with $x_i(t) = x + t(x_i - x)$ for $i = 1, ..., n$; attraction if $u\big(E(q(t))\big) \leqslant u(q(t))$; and neutrality if $u\big(E(q(t))\big) = u\big(q(t)\big)$. Consequently, if Assumption 2 holds, there is local risk aversion if

$$\lim_{t \to 0} \frac{1}{dt} \left(u(E(q(t))) - u\big(q(t)\big)\right)$$

is positive and only if it is nonnegative for all $x \in X$ and $q \in Q$; attraction if it is negative and only if it is nonpositive; and neutrality only if it is equal to zero.

Proposition 4. If Assumption 2 holds, $\langle Q, \succsim \rangle$ exhibits local risk aversion if $EV(x, q) > MEV(x, q)$ and only if $EV(x, q) \geqslant MEV(x, q)$ for all $x \in X$, $q \in Q_n$ and $n = 1, ..., m$ (where the local Expected Value function $EV(x, q)$ is introduced by Definition 3 and the local Marginal Expected Value is introduced by Definition 6); attraction if $EV(x, q) < MEV(x, q)$ and only if $EV(x, q) \leqslant MEV(x, q)$; and neutrality if and only if $EV(x, q) = MEV(x, q)$.

Proof: For every $x \in X$ and $q \in Q_n$ we have, taking into account Definitions 2, 3 and 4,

$$\text{(i)}\quad \lim_{t \to 0} \frac{\partial u(q(t))}{\partial x_i^h(t)} = p_i^h(x,\ q)\ \frac{\partial u(x)}{\partial x^h};$$

$$\text{(ii)}\quad \frac{du\big(E(q(t))\big)}{dt} = \sum_{h=1}^{k} \frac{\partial u\big(E(q(t))\big)}{\partial E^h\big(q(t)\big)} \sum_{i=1}^{n} (x_i^h - x^h)p_i;$$

$$\text{(iii)}\quad \frac{du\big(q(t)\big)}{dt} = \sum_{h=1}^{k} \sum_{i=1}^{n} \frac{\partial u\big(q(t)\big)}{\partial x_i^h(t)} (x_i^h - x^h);$$

and, consequently, since $E\big(q(t)\big) = (1 - t)x + tE(q)$,

$$\lim_{t \to 0} \frac{du\big(E(q(t))\big)}{dt},\ \sum_{h=1}^{k} \frac{\partial u(x)}{\partial x^h} \sum_{i=1}^{n} (x_i^h - x^h)p_i$$

and

$$\lim_{t \to 0} \frac{du\big(q(t)\big)}{dt} = \sum_{h=1}^{k} \sum_{i=1}^{n} p_i^h(x,\ q)\ \frac{\partial u(x)}{\partial x^h}\ (x_i^h - x^h)$$

so that

$$\lim_{t \to 0} \frac{d}{dt}\left(u(E(q(t))) - u\big(q(t)\big)\right) = \sum_{h=1}^{k} \frac{\partial u(x)}{\partial x_h} \sum_{i=1}^{n} \big(p_i - p_i^h\ (x, q)\big)\, x_i^h =$$

$$= \sum_{h=1}^{k} r^h\ (x)\big(E^h(q) - ME^h(x,\ q)\big) \sum_{s=1}^{k} \frac{\partial u(x)}{\partial x^s} =$$

$$= \big(EV(x,\ q) - MEV(x,\ q)\big) \sum_{s=1}^{k} \frac{\partial u(x)}{\partial x^s} \qquad \square$$

Proposition 5. If Assumption 2 holds, $\langle Q, \succsim \rangle$ exhibits global risk aversion if $EV\big(E(q), q\big) \geqslant MEV\big(E(q), q\big)$ for all $q \in Q_n$ and $n = 1, ..., m$, and $u : Q \to \mathbb{R}$ is a concave function of $(x_i)_{i=1}^n$ (i.e., the preference system is convex with respect to $(x_i)_{i=1}^n$ and the corresponding quasiconcave ordinal utility function is concavifiable); risk attraction if $EV\big(E(q),\ q\big) \leqslant MEV\big(E(q),\ q\big)$ and u is convex; and risk neutrality if $EV\big(E(q), q\big) = MEV\big(E(q), q\big)$ and u is linear.

Proof: Let us take under consideration the function $RP_u\ (t; q) = \big(u(E(q(t))) - u\big(q(t)\big)\big)$ for $t \in (0, 1]$ where $q(t) = (x_i(t), p_i)_{i=1}^n$ and $x_i(t) = tx_i + (1 - t)E(q)$. If $\lim_{t \to 0} RP_u(t;\ q) = 0$, $\lim_{t \to 0} \frac{dRP_u(t;\ q)}{dt} \geqslant 0$, and $RP_u(t;\ q) \leqslant tRP_u(1;\ q)$

for $t \in (0, 1]$, then $RP_u(1; q) \geqslant 0$, so that $u(E(q)) \geqslant u(q)$. It is $\lim_{t \to 0} RP_u(t; q) = 0$ since $\lim_{t \to 0} u(q(t)) = u(E(q))$ and $E(q(t)) = E(q)$ for $t \in (0, 1]$. It is $\lim_{t \to 0} \frac{dRP_u(t;\ q)}{dt} \geqslant 0$ since $\lim_{t \to 0} \frac{dRP_u(t;\ q)}{dt} = -\lim_{t \to 0} \frac{du(q(t))}{dt} = -\lim_{t \to 0} \Sigma_{h=1}^k \Sigma_{i=1}^n \frac{\partial u(q(t))}{\partial x_i^h(t)} (x_i^h - E^h(q)) = -\Sigma_{h=1}^k \Sigma_{i=1}^n p_i^h(E(q), q) \frac{\partial u(E(q))}{\partial E^h(q)} (x_i^h - E^h(q)) = -\Sigma_{h=1}^k r^h (E(q)) \Sigma_{i=1}^n p_i^h (E(q), q) (x_i^h - E^h(q)) \Sigma_{s=1}^k \frac{\partial u(E(q))}{\partial E^s(q)} = -(MEV(E(q), q) - EV(E(q), q)) \Sigma_{s=1}^k \frac{\partial u(E(q))}{\partial E^s(q)} \geqslant 0$ by assumption. Finally, since $u : Q \to \mathbb{R}$ is a concave function of $(x_i)_{i=1}^n$ then $RP_u(t; q) \leqslant tRP_u(1; q)$ for $t \in (0, 1]$ since $RP_u(t; q) - tRP_u(1; q) = u(E(q(t))) - u(q(t)) - tu(E(q)) + tu(q) = -u(q(t)) + tu(q) + (1 - t) u(E(q)) \leqslant 0$. □

Proposition 6. $\langle Q, \succsim \rangle$ exhibits global risk aversion if $\frac{u(q)}{u(E(q))}$ is a convex function of $(p_i)_{i=1}^n$ where $u : Q \to \mathbb{R}$ is a utility function which is positive for every degenerate lottery, i.e., $u(E(q)) > 0$ for every $E(q)$[1]; attraction if it is concave; and neutrality if $u(q) = u(E(q))$.

Proof: Let us take into consideration for every $(x_i)_{i=1}^n$ the function $\phi((p_i)_{i=1}^n; (x_i)_{i=1}^n) = \frac{u(q)}{u(E(q))} - 1$, which is convex by assumption. Consequently, the set $\Phi((x_i)_{i=1}^n) = \{(p_i)_{i=1}^n : \phi((p_i)_{i=1}^n; (x_i)_{i=1}^n) \leqslant 0\}$ is convex. We find that all vertices of the probability simplex $\{(p_i)_{i=1}^n \in \mathbb{R}_+^n : \Sigma_{i=1}^n p_i = 1\}$ belong to $\Phi((x_i)_{i=1}^n)$, since for $p_i = 1$ we have $u(E(q)) = u(x_i)$ for $i = 1, ..., n$. Since all points of the simplex are linear convex combinations of the vertices and since these vertices belong to the set $\Phi((x_i)_{i=1}^n)$, which is convex, then $\Phi((x_i)_{i=1}^n)$ coincides with the probability simplex. Consequently, for every $(x_i)_{i=1}^n$ we have $\phi((p_i)_{i=1}^n) \leqslant 0$ for all $(p_i)_{i=1}^n$, i.e., $\frac{u(q)}{u(E(q))} - 1 \leqslant 0$ and, therefore, $u(q) \leqslant u(E(q))$ for all $q \in Q$. □

3.5. Aversion to increasing risk

Proposition 7. $\langle Q, \succsim \rangle$ exhibits attraction to MPAS-decreasing risk (introduced by Definition 15) if and only if $MPAS(q^*) \subseteq \bar{G}_Q^C(u(q^*))$ (these sets are introduced by Definition 8 and 9) for all $q^* \in Q$; aversion if and only if $MPAS(q^*) \subseteq G_Q(u(q^*))$; and neutrality if and only if $MPAS(q^*) \subseteq (\bar{G}_Q^C(u(q^*)) \cap G_Q(u(q^*)))$.

[1] If $u : Q \to \mathbb{R}$ does not indicate a positive utility for every degenerate lottery, then we can represent the preference system $\langle Q, \succsim \rangle$ through the utility function $u' : Q \to \mathbb{R}$, with $u'(q) = u(q) + 1 + \max_{\hat{q} \in Q} |u(\hat{q})|$, so that $u'(q) > 0$ for every $q \in Q$.

Proof: Let us first demonstrate the necessary condition. Since $u(q) \geqslant u(q^*)$ is equivalent to $q \in \bar{G}_Q^C(u(q^*))$, then $u(q) \geqslant u(q^*)$ for all $q \in MPAS(q^*)$ implies $MPAS(q^*) \subseteq \bar{G}_Q^C(u(q^*))$. Let us now demonstrate the sufficient condition. $MPAS(q^*) \subseteq \bar{G}_Q^C(u(q^*))$ implies that if $q \in MPAS(q^*)$ then $q \in \bar{G}_Q^C(u(q^*))$, i.e., $u(q) \geqslant u(q^*)$. □

Graphic representation: In the Marschak-Machina diagram, with $x_1 > x_3$ and $x_2 = \lambda x_1 + (1 - \lambda)x_3$ with $\lambda \in (0, 1)$, the set $MPAS(q^*)$, where $q^* = (x_i, p_i^*)_{i=1}^3$, is that portion of the ser $\{q : E(q) = E(q^*)\}$ which is southwest with respet to point q^*. Proposition 7 requires for attraction that $MPAS(q^*) \subseteq \bar{G}_Q^C(u(q^*))$, i.e., that the indifference curve $q \sim q^*$ be upper than $MPAS(q^*)$, as represented in Figure 2. Note that $MPAS(q^*)$ is a segment of the line $p_c = \frac{1 - \lambda}{\lambda} (p_a - p_a^*) + p_c^*$.

Assumption 3. The utility function $u : Q \to \mathbb{R}$ is differentiable with respect to $(p_i)_{i=1}^n$

Proposition 8. If Assumption 3 holds, $\langle Q, \succsim \rangle$ exhibits attraction to MPAS-decreasing risk if and only if $\frac{\partial u(q)}{\partial p_b} \geqslant \lambda \frac{\partial u(q)}{\partial p_a} + (1 - \lambda) \frac{\partial u(q)}{\partial p_c}$ for all $q \in Q$, where x_a, x_b, x_c is every triplet $x_a, x_b, x_c \in X$ such that $x_b = \lambda x_a + (1 - \lambda)x_c$ with $\lambda \in (0, 1)$, i.e., if and only if the derivatives of $u(q)$ with respect to probabilities are a concave function of the corresponding linearly dependent consequences; aversion if and only if these derivatives are a convex function; and neutrality if and only if they are a linear function.

Proof: Let us first demostrate the necessary condition. The condition $u(q) \geqslant u(q^*)$ for all $q \in MPAS(q^*)$ and $q^* \in Q$ implies that $\left(\frac{du(q)}{dp_b}\right)_{MPAS} \geqslant 0$, where $\left(\frac{du(q)}{dp_b}\right)_{MPAS} -\lambda \frac{\partial u(q)}{\partial p_a} + \frac{\partial u(q)}{\partial p_b} - (1 - \lambda)\frac{\partial u(q)}{\partial p_c}$, since $x_b = \lambda x_a + (1 - \lambda)x_c$ for every x_a, $x_c \in X$ and $\lambda \in (0, 1)$, $x_a(p_a - p_a^*) + x_b(p_b - p_b^*) + x_c(p_c - p_c^*) = 0$ and $p_a - p_a^* + p_b - p_b^* + p_c -$

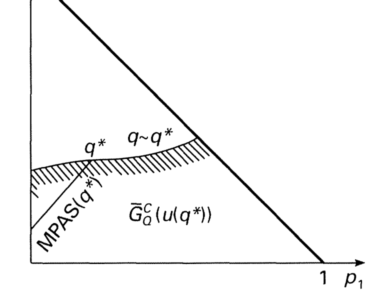

Figure 2

$p_c^* = 0$ (with $p_b - p_b^* > 0$) so that $p_a - p_a^* = -\lambda(p_b - p_b^*)$ and $p_c - p_c^* = -(1-\lambda)\,(p_b - p_b^*)$. The condition $\left(\frac{du(q)}{dp_b}\right)_{MPAS} \geqslant 0$ for all $q^* \in Q$ is, thus, equivalent to $\frac{\partial u(q)}{\partial p_b} \geqslant \lambda \frac{\partial u(q)}{\partial p_a} + (1-\lambda)\frac{\partial u(q)}{\partial p_c}$ for all $q^* \in Q$. Let us now demonstrate the sufficient condition. If it is not $u(q) \geqslant u(q^*)$ for all $q \in MPAS(q^*)$ and $q^* \in Q$, then there is a pair $q, q^* \in Q$ with $q \in MPAS(q^*)$ such that $u(q) < u(q^*)$. Then, introducing the lotteries $q(t) = (x_i, p_i^* + t(p_i - p_i^*))_{i=1}^n$ so that $q(0) = q^*$ and $q(1) = q$, we have that $u(q) < u(q^*)$ implies the existence of at least a $\hat{t} \in [0, 1]$ for which $\left.\frac{du(q(t))}{dt}\right|_{t=\hat{t}} < 0$, i.e., there is a $\hat{q} = q(\hat{t}) = (x_i, \hat{p}_i)_{i=1}^n$ for which (since $p_i^* = p_i = \hat{p}_i$ for every $i = 1, ..., n$ bar a, b, c) $\left.\frac{du(q(t))}{dt}\right|_{t=\hat{t}} = \frac{\partial u(\hat{q})}{\partial \hat{p}_a}(p_a - p_a^*) + \frac{\partial u(\hat{q})}{\partial \hat{p}_b}(p_b - p_b^*) + \frac{\partial u(\hat{q})}{\partial \hat{p}_c}(p_c - p_c^*) < 0$, which implies (since $p_a - p_a^* = -\lambda(p_b - p_b^*)$ and $p_c - p_c^* = -(1-\lambda)(p_b - p_b^*)$)

$$\left(-\lambda\frac{\partial u(\hat{q})}{\partial \hat{p}_a} + \frac{\partial u(\hat{q})}{\partial \hat{p}_b} - (1-\lambda)\frac{\partial u(\hat{q})}{\partial \hat{p}_c}\right)(p_b - p_b^*) < 0$$

i.e., since $p_b - p_b^* > 0$

$$\frac{\partial u(\hat{q})}{\partial \hat{p}_b} < \lambda\frac{\partial u(\hat{q})}{\partial \hat{p}_a} + (1-\lambda)\frac{\partial u(\hat{q})}{\partial \hat{p}_c}$$

Consequently, the condition

$$\frac{\partial u(q)}{\partial p_a} \geqslant \lambda\frac{\partial u(q)}{\partial p_b} + (1-\lambda)\frac{\partial u(q)}{\partial p_c}$$

for all $q \in Q$, where $x_a, x_c \in X$, $x_b = \lambda x_a + (1-\lambda)x_c$ and $\lambda \in (0, 1)$, implies $u(q) \geqslant u(q^*)$ for all $q \in MPAS(q^*)$ and $q^* \in Q$. □

Remark: The kind of concavity taken into account by Proposition 11 does not require function $u : Q \to \mathbb{R}$ be cardinal: if we introduce a monotonically increasing transformation $v : \mathbb{R} \to \mathbb{R}$ of u, i.e., $v = g(u)$ with $\frac{dg(u)}{du} > 0$, then $\frac{\partial v(q)}{\partial p_a} \geqslant \lambda\frac{\partial v(q)}{\partial p_b} + (1-\lambda)\frac{\partial v(q)}{\partial p_c}$ is equivalent to $\frac{dg(u)}{du}\frac{du(q)}{\partial p_b} \geqslant \frac{dg(u)}{du}\frac{du(q)}{\partial p_a} + (1-\lambda)\frac{dg(u)}{du}\frac{du(q)}{\partial p_c}$, i.e., to $\frac{\partial u(q)}{\partial p_b} \geqslant \lambda\frac{\partial u(q)}{\partial p_a} + (1-\lambda)\frac{\partial u(q)}{\partial p_c}$.

Graphic representation: Since $p_b = 1 - \Sigma_{i\neq b,\, i=1}^n p_i$, the condition:

$$\frac{\partial u(q)}{\partial p_b} \geqslant \lambda\frac{\partial u(q)}{\partial p_a} + (1-\lambda)\frac{\partial u(q)}{\partial p_c}$$

becomes, when p_b is introduced in $q = (x_i, p_i)_{i=1}^n$ as $1 - \Sigma_{i\neq b,\, i=1}^n p_i$ the condition:

$$0 \geqslant \lambda\frac{\partial u(q)}{\partial p_a} + (1-\lambda)\frac{\partial u(q)}{\partial p_c}$$

i.e., assuming that $\frac{\partial u(q)}{\partial p_c} < 0$,

$$\left(\frac{dp_c}{dp_a}\right)_{u(q)} = -\frac{\partial u(q)/\partial p_a}{\partial u(q)/\partial p_c} \leqslant \frac{1-\lambda}{\lambda} = \left(\frac{dp_c}{dp_a}\right)_{u(E(q))}$$

since

$$\left(\frac{dp_c}{dp_a}\right)_{u(E(q))} = -\frac{\partial u(E(q))/\partial p_a}{\partial u(E(q))/\partial p_c} = \frac{1-\lambda}{\lambda}$$

Taking into account that

$$\frac{\partial u(E(q))}{\partial p_a} = \frac{du(E(q))}{dE(q)}(x_a - x_b) \qquad \frac{\partial u(E(q))}{\partial p_c} = \frac{du(E(q))}{dE(q)}(x_c - x_b)$$

and $x_a - x_b = (1-\lambda)\,(x_a - x_c)$, $x_c - x_b = -\lambda(x_a - x_c)$.

In the Marschak-Machina diagram (with $x_b = \lambda x_a + (1-\lambda)x_c$), this condition requires that there be no point q^* in the diagram where the slope of the indifference curve $u(q) = u(q^*)$ exceeds the slope of the indifference curve $u(E(q)) = u(E(q^*))$, which coincides with the slope of $MPAS(q^*)$.

Proposition 9. $\langle Q, \succsim \rangle$ exhibits aversion to PM-increasing risk (introduced by Definition 16) if and only if the set $G_Q(u)$ (introduced by Definition 9) is convex with respect to probabilistic mixtures for all $u \in \mathbb{R}$, i.e., if and only if $q_a, q_b \in G_Q(u)$ implies $(\lambda q_a \oplus (1-\lambda)q_b) \in G_Q(u)$ for all $\lambda \in [0, 1]$; attraction if $\bar{G}_Q^C(u)$ is convex; and neutrality if and only if both $G_Q(u)$ and $\bar{G}_Q^C(u)$ are convex.

Proof: Let us first demonstrate the necessary condition. If there is a $G_Q(u)$ which is no convex with respect to probabilistic mixtures, then there are a $u \in \mathbb{R}$, a pair $q_a, q_b \in G_Q(u)$ and a $\lambda \in [0, 1]$ such that $\lambda q_a \oplus (1-\lambda)q_b \notin G_Q(u)$, i.e., $u(\lambda q_a \oplus (1-\lambda)q_b) > u$ while $u(q_a) \leqslant u$ and $u(q_b) \leqslant u$, so that $u(\lambda q_a \oplus (1-\lambda)q_b) > \max\{u_a, u_b\}$. Let us now demonstrate the sufficient condition. If $G_Q(u)$ is convex with respect to probabilistic mixtures, then for every pair $q_a, q_b \in Q$, since $q_a, q_b \in G_Q(\max\{u_a, u_b\})$, it is also $(\lambda q_a \oplus (1-\lambda)q_b) \in G_Q(\max\{u_a, u_b\})$ for all $\lambda \in [0, 1]$, i.e., $u(\lambda q_a \oplus (1-\lambda)q_b) \leqslant \max\{u(q_a), u(q_b)\}$ for all $q_a, q_b \in Q$, since $q_a, q_b \in Q$ and $\lambda \in [0, 1]$. □

3.6. Aversion to risk and to increasing risk when the Expected Utility model applies

If the Expected Utility model applies, i.e., if there is a function $U : X \to \mathbb{R}$ such that the lotteries $q \in Q$ are ordered according to the EU function $EU : Q \to \mathbb{R}$, with $EU(q) = \Sigma_{i=1}^{n} p_i U(x_i)$, where $q = (x_i, p_i)_{i=1}^{n}$, then the following propositions hold.

Proposition 10. When the EU model applies, then $\langle Q, \succsim \rangle$ exhibits risk aversion if and only if the utility function $U : X \to \mathbb{R}$ is concave; attraction if and only if it is convex; and neutrality if and only if it is linear.

Proposition 11. When the EU model applies, then $\langle Q, \succsim \rangle$ exhibits attraction to MPAS-decreasing risk if and only if the utility function $U : X \to \mathbb{R}$ is concave; attraction if and only if it is convex; and neutrality if and only if it is linear.

Proposition 12. When the EU model applies, then $\langle Q, \succsim \rangle$ exhibits neutrality to PM-increasing risk.

Proofs: Proposition 10 can be easily demonstrated taking into account that $U(\Sigma_{i=1}^{n} p_i x_i) - \Sigma_{i=1}^{n} p_i U(x_i)$ is nonnegative for all $q \in Q$ if and only if U is concave and that $U(\Sigma_{i=1}^{n} p_i x_i) = EU(E(q), 1)$. Proposition 11 is a corollary of Proposition 8 since $\dfrac{\partial EU(q)}{\partial p_i} = U(x_i)$. Proposition 12 is trivial since $EU : q \to \mathbb{R}$ is linear with respect to probabilities. □

3.7. Aversion to risk and to increasing risk when the Rank Dependent Expected Utility model applies

If the Rank Dependent Expected Utility model applies, i.e., if there are a utility function $U : X \to \mathbb{R}$ and a distortion function of probability $\phi : [0, 1] \to [0, 1]$, where ϕ is monotonically nondecreasing with $\phi(0) = 0$ and $\phi(1) = 1$, such that the lotteries $q \in Q$ are ordered according to the *RDEU* function $RDEU : Q \to \mathbb{R}$, with $RDEU(q) = U(x_n) + \Sigma_{i=1}^{n-1}(U(x_i) - U(x_{i+1}))\phi(\Sigma_{v=1}^{i} p_v)$, where $q = (x_i, p_i)_{i=1}^{n}$, with $U(x_i) \geqslant U(x_{i+1})$ for $i = 1, ..., n-1$, then the following propositions hold.

Proposition 13. (Montesano and Giovannoni, 1996). When the RDEU model applies, then $\langle Q, \succsim \rangle$ exhibits risk aversion of the first order if and only if the distortion function of probability satisfies the condition $\phi(p) \leqslant p$ for every $p \in [0, 1]$ (attraction if and only if $\phi(p) \geqslant p$, and neutrality if $\phi(p) = p$), and risk aversion of the second order if and only if the utility function is concave over X (attraction if and only if it is convex, and neutrality if and only if it is linear). (There is risk aversion of the first order if $EU(q) - RDEU(q)$ is nonnegative for every $q \in Q$, attaction if it is nonpositive and neutrality if it is equal to zero. There is risk aversion of the second order if $U(E(q)) - EU(q)$ is nonnegative for every $q \in Q$, attraction if it is nonpositive, and neutrality if it is equal to zero). Consequently, $\langle Q, \succsim \rangle$ exhibits risk aversion, i.e., $U(E(q)) - RDEU(q)$ is nonnegative for every $q \in Q$, if $\phi(p) \leqslant p$ and U is concave; attraction if $\phi(p) \geqslant p$ *and* U is convex; and neutrality if and only if $\phi(p) = p$ and U is linear.

Proof: The risk aversion of the second order is that kind of risk aversion already indicated for the EU model, so that the same condition applies. With regard to the risk avesion of the first order, since $EU(q) - RDEU(q) = \Sigma_{i=1}^{n-1}(U(x_i) - U(x_{i+1}))(\Sigma_{v=1}^{i} p_v - \phi(\Sigma_{v=1}^{i} p_v))$, it is easy to see that $p - \phi(p) \geqslant 0$ for every $p \in [0, 1]$ is a sufficient and necessary condition in order that $EU(q) - RDEU(q) \geqslant 0$ for every $q \in Q$. □

Proposition 14. When the RDEU model applies, then $\langle Q, \succsim \rangle$ exhibits attraction to MPAS-decreasing risk if the utility function is concave over X and the distortion function of probability is convex over [0, 1]; aversionl if U is convex and ϕ is concave; and neutrality if U and ϕ are linear.

Proof: Taking into account Definitions 8 and 15, there is attraction to MPAS-decreasing risk if $RDEU(q) \geqslant RDEU(q^*)$ for every pair $q, q^* \in Q$ with $q \in MPAS(q^*)$. If $U : X \to \mathbb{R}$ is concave, then $U(x_b) \geqslant \lambda U(x_a) + (1 - \lambda)U(x_c)$, where $x_b = \lambda x_a + (1 - \lambda)x_c$ and $p_a - p_a^* = -\lambda(p_b - p_b^*)$, $p_c - p_c^* = -(1 - \lambda)(p_b - p_b^*)$, and $p_b - p_b^* \geqslant 0$. Without any loss of generality let $U(x_a) \geqslant U(x_c)$. If $U(x_b) \geqslant U(x_a)$, then $RDEU(q) - RDEU(q^*) = \Sigma_{i=b}^{a-1}(U(x_i) - U(x_{i+1}))(\phi(\Sigma_{v=1}^{i} p_v^* + p_b - p_b^*) - \phi(\Sigma_{v=1}^{i} p_v^*)) + \Sigma_{i=a}^{c-1}(U(x_i) - U(x_{i+1}))(\phi(\Sigma_{v=1}^{i} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{i} p_v^*)) \geqslant 0$ since ϕ is a monotonically nondecreasing function and $U(x_i) - U(x_{i+1}) \geqslant 0$ for $i = 1, ..., n-1$ (the first addendum in the expression of $RDEU(q) - RDEU(q^*)$ vanishes if $U(x_b) = U(x_a)$, the second does if $U(x_a) = U(x_c)$). If $U(x_c) < U(x_b) < U(x_a)$, then $RDEU(q) - RDEU(q^*) = \Sigma_{i=a}^{b-1}(U(x_i) - U(x_{i+1}))(\phi(\Sigma_{v=1}^{i} p_v^* - \lambda(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{i} p_v^*)) + \Sigma_{i=b}^{c-1}(U(x_i) - U(x_{i+1}))(\phi(\Sigma_{v=1}^{i} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{i} p_v^*))$. Since ϕ is convex, then $\phi(\Sigma_{v=1}^{i} p_v^* - \lambda(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{i} p_v^*)) \geqslant \phi(\Sigma_{v=1}^{b-1} p_v^* - \lambda(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{b-1} p_v^*)$ for $a \leqslant i \leqslant b-1$; $\phi(\Sigma_{v=1}^{i} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{i} p_v^*)) \geqslant \phi(\Sigma_{v=1}^{b} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{b} p_v^*)$ for $i \geqslant b$; and $\phi(\Sigma_{v=1}^{b} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{b} p_v^*)) \geqslant \dfrac{1 - \lambda}{\lambda}(\phi(\Sigma_{v=1}^{b-1} p_v^*) - \phi(\Sigma_{v=1}^{b-1} p_v^* - \lambda(p_b - p_b^*)))$. Moreover, since U is concave, then $U(x_b) - U(x_c) \geqslant \dfrac{\lambda}{1 - \lambda}(U(x_a) - U(x_b))$. Therefore, $RDEU(q) - RDEU(q^*) \geqslant (U(x_a) - U(x_b))(\phi(\Sigma_{v=1}^{b-1} p_v^* - \lambda(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{b-1} p_v^*)) + (U(x_b) - U(x_c))\phi(\Sigma_{v=1}^{b} p_v^* + (1 - \lambda)(p_b - p_b^*)) - \phi(\Sigma_{v=1}^{b} p_v^*)) \geqslant 0$. □

Proposition 15. (Montesano and Giovannoni, 1996, pp. 141-142). When the RDEU model applies, then $\langle Q, \succsim \rangle$ exhibits aversion to PM-increasing risk (introduced by Definition 16) if and only if the distortion function of probability is convex.

Proof: If ϕ is convex, then $RDEU(\lambda q_a \oplus (1-\lambda)q_b) = U(x_n) + \Sigma_{i=1}^{n-1} \big(U(x_i) - U(x_{i+1})\big)\phi\big(\Sigma_{v=1}^{i} \lambda p_v^a + (1-\lambda)p_v^b\big) \leqslant U(x_n) + \Sigma_{i=1}^{n-1} \big(U(x_i) - U(x_{i-1})\big)\big(\lambda\phi(\Sigma_{v=1}^{i} p_v^a + (1-\lambda)\phi(\Sigma_{v=1}^{i} p_v^b)\big) = \lambda RDEU(q_a) + (1-\lambda)RDEU(q_b) \leqslant \max\{RDEU(q_A), RDEU(q_b)\}$, where $q_a = (x_i, p_i^a)_{i=1}^n$, $q_b = (x_i, p_i^b)_{i=1}^n$, $\lambda q_a \oplus (1-\lambda)q_b = (x_i, \lambda p_i^a + (1-\lambda)p_i^b)_{i=1}^n$, and $U(x_i) \geqslant U(x_{i+1})$ for $i = 1, ..., n-1$. If ϕ is not convex, then there exist three probabilities p_1, p_2, p_3 with $p_1 > p_2 > p_3$ such that $\phi\left(\frac{p_1 + p_2}{2}\right) \geqslant \frac{1}{2}(\phi(p_1) + \phi(p_2))$ and $\phi\left(\frac{p_2 + p_3}{2}\right) \geqslant \frac{1}{2}(\phi(p_2) + \phi(p_3))$, where at least one of these two inequalities is strong. Let the following two lotteries be taken into consideration: $q_a = (x_1, p_2; x_2, 0; x_3, 1 - p_2)$, and $q_b = (x_1, p_3; x_2, p_1 - p_3; x_3, 1 - p_1)$, where $U(x_1) > U(x_2) > U(x_3)$ and $\frac{U(x_2) - U(x_3)}{U(x_1) - U(x_2)} = \frac{\phi(p_2) - \phi(p_3)}{\phi(p_1) - \phi(p_2)}$. We find that $RDEU(q_a) = RDEU(q_b)$ since $RDEU(q_a) - RDEU(q_b) = \big(U(x_1) - U(x_2)\big)\big(\phi(p_2) - \phi(p_3)\big) + \big(U(x_2) - U(x_3)\big)\big(\phi(p_2) - \phi(p_1)\big) = 0$. Moreover, $RDEU(0.5q_a \oplus 0.5q_b) = U(x_3) + (U(x_1) - U(x_2))\phi\left(\frac{p_2 + p_3}{2}\right) + \big(U(x_2) - U(x_3)\big)\phi\left(\frac{p_1 + p_2}{2}\right) > U(x_3) + \frac{1}{2}\big(U(x_1) - U(x_2)\big)\big(\phi(p_2) + \phi(p_3)\big) + \frac{1}{2}\big(U(x_2) - U(x_3)\big)\big(\phi(p_1) + \phi(p_2)\big) = \frac{1}{2}RDEU(q_a) + \frac{1}{2}RDEU(q_b) = \max\{RDEU(q_a), RDEU(q_b)\}$ since $RDEU(q_a) = RDEU(q_b)$. Consequently, if ϕ is not convex, then it is not $RDEU(\lambda q_a \oplus (1-\lambda)q_b) \leqslant \max\{RDEU(q_a), RDEU(q_b)\}$ for every pair q_a, $q_b \in Q$ and $\lambda \in [0, 1]$. □

4. UNCERTAINTY AND RISK AVERSION IN A DECISION MAKING UNDER UNCERTAINTY SITUATION

4.1. Definition of global risk and uncertainty aversion and aversion to increasing risk and uncertainty

With respect to a DMUR situation, when DMUU situation is taken into consideration there is a further reason for possible aversion in the preference system, which is called uncertainty (or ambiguity) aversion and is related to preference for knowing the chances.

Definition 19. (Global risk & uncertainty aversion): $\langle F, \succsim \rangle$ exhibits risk & uncertainty aversion if there is a $p^* \in P$ such that $u\big(E(f, p^*)\big) \geqslant u(f)$ for all $f \in F$; risk & uncertainty attraction if there ia a $p^* \in$ P such that $u\big(E(f, p^*)\big) \leqslant u(f)$ for all $f \in F$; and risk & uncertainty neutrality if there is a $p^* \in P$ such that $u\big(E(f, p^*)\big) = u(f)$ for all $f \in F$.

The following Proposition states that the only possibility for both aversion and attraction to risk & uncertainty is given by neutrality.

Proposition 16. There no pair p', $p'' \in P$ with $p' \neq p''$ such that $u\big(E(f, p')\big) \geqslant u(f) \geqslant u\big(E(f, p'')\big)$ for all $f \in F$ if there is a pair x', $x'' \in X$ such that $u(x') > u(x'')$.

Proof: Let $x(t) = tx'' + (1-t)x'$, then $x(t) \in X$ for $t \in [0, 1]$ since X is a convex set. The continuity of the preference system (implied by the existence of an ordinal utility function) requires that there be a pair t', $t'' \in [0, 1]$ with $t' < t''$ such that $u\big(x(t_1)\big) > u\big(x(t_2)\big)$ for every pair t_1, $t_2 \in [t', t'']$ with $t_1 < t_2$. For every pair p', $p'' \in P$ with $p' \neq p''$ there is at least a state of nature $s^* \in S$ for which $p'(s^*) < p''(s^*)$. Consequently, for an act $f = \big(x(t'), s^*; x(t''), S\backslash\{s^*\}\big)$ we have $E(f, p') = \big(t' + (t'' - t')(1 - p'(s^*))\big)x'' + \big(1 - t' - (t'' - t')(1 - p'(s^*))\big)x'$ and $E(f, p'') = \big(t' + (t'' - t')(1 - p''(s^*))\big)x'' + \big(1 - t' - (t'' - t')(1 - p''(s^*))\big)x'$, so that, since $t' < t' + \big(t'' - t')(1 - p''(s^*)\big) < t' + (t'' - t')(1 - p'(s^*)\big) < t''$, we find $u\big(E(f, p')\big) < u\big(E(f, p'')\big)$. Therefore, there is no pair p', $p'' \in P$ with $p' \neq p''$ such that $u\big(E(f, p')\big) \geqslant u\big(E(f, p'')\big)$ for all $f \in F$ and, a fortiori, such that $u\big(E(f, p')\big) \geqslant u(f) \geqslant u\big(E(f, p'')\big)$ for all $f \in F$. □

Definition 20. (Global uncertainty aversion): $\langle F, \succsim \rangle$ exhibits uncertainty aversion if there is a $p^* \in P$ such that $u(f, p^*) \geqslant u(f)$ for all $f \in F$ (where (f, p^*) is the lottery $\big(x_i, p^*(e_i)\big)_{i=1}^n$ induced through the act $f = (x_i, e_i)_{i=1}^n$ by the probability distribution p^*); uncertainty attraction if there is a $p^* \in P$ such that $u(f, p^*) \leqslant u(f)$ for all $f \in F$; and uncertainty neutrality if there is a $p^* \in P$ such that $u(f, p^*) = u(f)$ for all $f \in F$.

Proposition 17. There is no pair p', $p'' \in P$ with $p' \neq p''$ such that $u(f, p') \geqslant u(f) \geqslant u(f, p'')$ for all $f \in F$, if lotteries with more probable best consequences are preferred (i.e., $u(x') > u(x'')$ and $p''(s) > p'(s)$ imply $u(q'') > u(q')$, where $q = \big(x(s), p(s); x(S\backslash\{s\}), 1 - p(s)\big)$ with $x(s) = x'$, $x(S\backslash\{s\}) = x''$ and $q' = \big(x', p'(s); x'', 1 - p'(s)\big)$, $q'' = \big(x', p''(s); x'', 1 - p''(s)\big)$.

Proof: For every pair p', $p'' \in P$ with $p' \neq p''$ there is at least a state of nature $s^* \in S$ for which $p'(s^*) < p''(s^*)$. Consequently, for an act $f = (x', s^*; x'', S\backslash\{s^*\})$ with $u(x'') > u(x'')$ we have $u(f, p') < u(f, p'')$. Therefore, there is no pair p', $p'' \in P$ with $p' \neq p''$ such that $u(f, p') \geqslant u(f, p'')$ for all $f \in F$ and, a fortiori, such that $u(f, p') \geqslant u(f) \geqslant u(f, p'')$ for all $f \in F$. □

Definition 21. (Global risk aversion): $\langle F, \succsim \rangle$ exhibits risk aversion if $u\big(E(f, p)\big) \geqslant u(f, p)$ for all $f \in F$ and $p \in P$; risk attraction if $u\big(E(f, p)\big) \leqslant u(f, p)$; and risk neutrality if $u\big(E(f, p)\big) = u(f, p)$.

Remarks: (i) Definition 21 coincides with Definition 14.

(ii) If $\langle F, \succsim \rangle$ exhibits both uncertainty aversion (attraction) and risk aversion (attraction), then $\langle F, \succsim \rangle$ also exhibits risk & uncertainty aversion (attraction). If $\langle F, \gtrsim \rangle$ exhibits both risk & uncertainty aversion (attraction) and risk attraction (aversion), then $\langle F, \gtrsim \rangle$ also exhibits uncertainty aversion (attraction). If $\langle F, \gtrsim \rangle$ exhibits risk & uncertainty aversion (attraction) and uncertainty attraction (aversion), then $\langle F, \gtrsim \rangle$ does not necessarily exhibit risk aversion (attraction).

Comparative risk and uncertainty aversion, which can be introduced analogously to comparative risk aversion (see Definition 17), suffers the same drawbacks as comparative risk aversion (see Proposition 1) and, consequenty, it is waived in the present paper.

Definition 22. (Aversion to increasing uncertainty & PM-decreasing risk): $\langle F, \gtrsim \rangle$ exhibits aversion to increasing uncertainty & PM-decreasing risk if $u(\lambda f_a \oplus (1-\lambda) f_b) \geqslant \min\{u(f_a), u(f_b)\}$ for all $f_a, f_b \in F$ and $\lambda \in [0, 1]$; attraction if $u(\lambda f_a \oplus (1-\lambda) f_b) \leqslant \max\{u(f_a), u(f_b)\}$; and neutralityl if both.

Remark: This Definition depends on the observation that a probabilistic mixture increases risk (Wakker, 1994) and reduces uncertainty (Schmeidler, 1989).

Definition 23. (Aversion to PM-increasing risk): $\langle F, \gtrsim \rangle$ exhibits aversion to PM-increasing risk if $u(\lambda f_a \oplus (1-\lambda) f_b, p) \leqslant \max\{u(f_a, p), u(f_b, p)\}$ for all $f_a, f_b \in F$, $p \in P$ and $\lambda \in [0, 1]$; attraction if $u(\lambda f_a \oplus (1-\lambda) f_b, p) \geqslant \min\{u(f_a, p), u(f_b, p)\}$; and neutrality if both.

Remarks: (i) Definition 23 coincides with Definition 16.

(ii) If there is neutrality to PM-increasing risk, then Definition 22 concerns aversion (attraction, neutrality) to increasing uncertainty.

4.2. Some Propositions on global risk and uncertainty aversion

Proposition 18. $\langle F, \gtrsim \rangle$ exhibits risk & uncertainty aversion (introduced by Definition 19) if and only if there is a $p^* \in P$ such that $H_F(u, p^*) \subseteq G_F(u)$ (these sets are introduced by Definitions 11 and 12) for all $u \in \mathbb{R}$; attraction if and only if $H_F(u, p^*) \supseteq G_F(u)$; and neutrality if and only if $H_F(u, p^*) = G_F(u)$.

Proof: Let us first demonstrate the necessary condition. If there is no $p \in P$ such that $H_F(u, p) \subseteq G_F(u)$ for all $u \in \mathbb{R}$, then for every pair $u \in \mathbb{R}$ and $p \in P$ there is a $f \in H_F(u, p)$ such that $f \notin G_F(u)$, i.e., $u(E(f, p)) \leqslant u$ and $u(f) > u$, so that $u(E(f, p)) < u(f)$. Therefore, for every $p \in P$ there is a $f \in F$ such that $u(E(f, p)) < u(f)$, i.e., $\langle F, \gtrsim \rangle$ does not exhibit risk & uncertainty aversion. Let us now demostrate the sufficient condition. If there is a $p^* \in P$ such that $H_F(u, p^*) \subseteq G_F(u)$ for all $u \in \mathbb{R}$, since $f \in H_F(u(E(f, p), p)$ for all $f \in F$ and $p \in P$, then it is also $f \in G_F(u(E(f, p^*)))$, i.e., $u(f) \leqslant u(E(f, p^*))$ for all $f \in F$. □

Proposition 19. $\langle F, \gtrsim \rangle$ exhibits uncertainty aversion (introduced by Definition 20) if and only if there is a $p^* \in P$ such that $L_F(u, p^*) \subseteq G_F(u)$ (these sets are introduced by Definitions 11 and 13) for all $u \in \mathbb{R}$; attraction if and only if $L_F(u, p^*) \supseteq G_F(u)$; and neutrality if and only if $L_F(u, p^*) = G_F(u)$.

Proof: Analogous to the proof of Proposition 18, taking $L_F(u, p)$ and $u(f, p)$ respectively in place of $H_F(u, p)$ and $u(E(f, p))$. □

Proposition 20. $\langle F, \gtrsim \rangle$ exhibits risk aversion if and only if $H_F(u, p) \subseteq L_F(u, p)$ (these sets are introduced by Definitions 12 and 13) for all pairs $u \in \mathbb{R}$ and $p \in P$; attraction if and only if $H_F(u, p) \supseteq L_F(u, p)$; and neutrality if and only if $H_F(u, p) = L_F(u, p)$.

Proof: Analogous to the proof of Proposition 2, taking (f, p), H_F and L_F respectively in place of q, H_Q and G_Q. □

Proposition 21. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits risk & uncertainty aversion if there is a $p^* \in P$ such that $EV(f, p^*) \geqslant MEV(f)$ (these functions are introduced by Definitions 3 and 6) for all $f \in F_n$ and $n = 1, ..., m$; attraction if $EV(f, p^*) \leqslant MEV(f)$; and neutrality if $EV(f, p^*) = MEV(f)$.

Proof: Let us introduce for every $f \in F_n$, $p \in P$ and $t \in (0, 1]$ the act $f(t) = (x_i(t), e_i)_{i=1}^n$, where $x_i(t) = tx_i + (1-t)E(f, p)$, and the utility & risk uncertainty premium function $RUP_u(t; f, p) = u(E(f(t), p)) - u(f(t))$. We find that $E(f(t), p) = E(f, p)$ for all $t \in (0, 1]$ and $\dfrac{dRUP_u(t; f, p)}{dt} = -\dfrac{du(f(t))}{dt} = -\Sigma_{h=1}^k \Sigma_{i=1}^n \dfrac{\partial u(f(t))}{\partial x_i^h(t)} (x_i^h - E^h(f, p)) = -\Sigma_{h=1}^k r^h(f(t)) \Sigma_{i=1}^n p_i^h(f(t))(x_i^h - E^h(f, p)) \Sigma_{s=1}^k \Sigma_{v=1}^n \dfrac{\partial u(f(t))}{\partial x_v^s(t)} = \dfrac{1}{t}(EV(f(t), p) - MEV(f(t))) \Sigma_{s=1}^k \Sigma_{v=1}^n \dfrac{\partial u(f(t))}{\partial x_v^s(t)}$, since $ME^h(f(t)) = \Sigma_{i=1}^n p_i^h(f(t))x_i^h(t) = t\Sigma_{i=1}^n p_i^h(f(t))x_i^h + (1-t)E^h(f, p)$ so that $\dfrac{1}{t}(E^h(f, p) - ME^h(f(t))) = ME^h(f(p)) - \Sigma_{i=1}^n p_i^h(f(t))x_i^h = -\Sigma_{i=1}^n p_i^h(f(t))(x_i^h - E^h(f, p))$ and $\dfrac{1}{t}(EV(f(t), p) - MEV(f(t))) = \dfrac{1}{t}\Sigma_{h=1}^k r^h(f(t))(E^h(f, p) - ME^h(f(t))) = -\Sigma_{h=1}^k r^h(f(t))\Sigma_{i=1}^n p_i^h(f(t))(x_i^h - E^h(f, p))$. Consequently, if there is a $p^* \in P$ such that $EV(f, p^*) \geqslant MEV(f)$ for all $f \in F_n$ and $n = 1, ..., m$, then $EV(f(t), p^*) \geqslant MEV(f(t))$ for all $t \in (0, 1]$ since $f(t) \in F_n$ for all $t \in (0, 1]$, so that $\dfrac{dRUP_u(t; f, p^*)}{dt} \geqslant 0$ for all $t \in (0, 1]$. Thus, since $\lim_{t \to 0}$

$RUP_u(t;\ f,\ p^*) = 0$ and $\frac{dRUP_u(t;\ f,\ p^*)}{dt} \geqslant 0$ for all $t \in (0, 1]$, then $RUP_u(1;\ f,\ p^*) \geqslant 0$, i.e., there is a $p^* \in P$ such that $u(E(f,\ p^*)) \geqslant u(f)$ for all $f \in F_n$ and $n = 1, ..., m$. □

Proposition 22. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits risk aversion if $EV(f, p) \geqslant MEV(f, p)$ (these functions are introduced by Definitions 3 and 6) for all $f \in F_n$ and $n = 1, ..., m$, and $p \in P$; attraction if $EV(f, p) \leqslant MEV(f, p)$; and neutrality if $EV(f, p) = MEV(f, p)$.

Proof: Analogous to the proof of Proposition 3 taking into account that (f, p) is a lottery.

Proposition 23. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits uncertainty aversion if $MEV(f, p) \geqslant EV(f, p)$ for all $f \in F_n$, $n = 1, ..., m$, and $p \in P$ and there is a $p^* \in P$ such that $EV(f, p^*) \geqslant MEV(f)$ for all $f \in F_n$ and $n = 1, ..., m$; attraction if $MEV(f, p) \leqslant EV(f, p)$ and $EV(f, p^*) \leqslant MEV(f)$; and neutrality if $MEV(f, p) = EV(f, p)$ and $EV(f, p^*) \geqslant MEV(f)$.

Proof: This Proposition clearly follows from Proposition 21 and 22, taking into account that risk & uncertainty aversion and risk attraction imply uncertainty aversion. □

4.3. Local risk and uncertainty aversion and other propositions on global risk and uncertainty aversion

Definition 24. (Local risk & uncertainty aversion): $\langle F, \gtrsim \rangle$ exhibits local risk & uncertainty aversion if for every $x \in X$ and $f \in F$ there are a $p^* \in P$ and a $t^* > 0$ such that $u(E(f(t), p^*)) \geqslant u(f(t))$ for every $t \in [0, t^*]$, where $f(t) = (x_i(t), e_i)_{i=1}^n$ with $x_i(t) = x + t(x_i - x)$ for $i = 1, ..., n$; attraction if $u(E(f(t), p^*)) \leqslant u(f(t))$; and neutrality if $u(E(f(t), p^*)) = u(f(t))$. Consequently, if Assumption 2 holds, there is local risk & uncertainty aversion if there is a $p^* \in P$ such that

$$\lim_{t \to 0} \frac{d}{dt} \left(u(f(t), p^*)) - u(f(t))\right)$$

is positive, and only if it is nonnegative, for all $x \in X$ and $f \in F$; attraction if it is negative, and only if it is nonpositive; and neutrality if and only if it is equal to zero.

Proposition 24. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits local risk & uncertainty aversion if there is a $p^* \in P$ such that $EV(x, f, p^*) - MEV(x, f)$ is positive (these functions are introduced by Definitions 3 and 6), and only if it is nonnegative for all $x \in X, f \in F_n$ and $n = 1, ..., m$; attraction if $EV(x, f, p^*) - MEV(x, f)$ is negative and only if it is nonpositive; and neutrality if and only if $EV(x, f, p^*) =$ $= MEV(x, f)$.

Proof: Analogous to the proof of Proposition 4, with f, F_n, p^*, $E(f, p^*)$ and $EV(x, f, p^*)$ respectively in place of q, Q_n, p, $E(q)$ and $EV(x, q)$. □

Proposition 25. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits global risk & uncertainty aversion if there is a $p^* \in P$ such that $EV(E(f, p^*), f, p^*) \geqslant MEV(E(f, p^*), f)$ for all $f \in F_n$ and $n = 1, ..., m$, and $u : F \to \mathbb{R}$ is a concave function of $(x_i)_{i=1}^n$; attraction if $EV(E(f, p^*), f, p^*) \leqslant MEV(E(f, p^*), f)$ and u is convex; and neutrality if $EV(E(f, p^*), f, p^*) = MEV(E(f, p^*), f)$ and u is linear.

Proof: Analogous to the proof of Proposition 5, with f, F, p^*, $E(f, p^*)$, $RUP_u(t;\ f,\ p^*)$, $f(t) = (x_i(t), e_i)_{i=1}^n$ and $EV(E(f, p^*), f, p^*)$ respectively in place of q, Q, p, $E(q)$, $RP_u(t;\ q)$, $q(t) = (x_i(t), p_i)_{i=1}^n$ and $EV(E(q), q)$. □

Definition 25. (Local uncertainty aversion): $\langle F, \gtrsim \rangle$ exhibits local uncertainty aversion if for every $x \in X$ and $f \in F$ there a $p^* \in P$ and a $t^* > 0$ such that $u(f(t), p^*) \geqslant u(f(t))$ for every $t \in [0, t^*]$, where $f(t) = (x_i(t), e_i)_{i=1}^n$ and $(f(t), p^*) = (x_i(t), p^*(e_i))_{i=1}^n$ with $x_i(t) = x + t(x_i - x)$ for $i = 1, ..., n$; attraction if $u(f(t), p^*) \leqslant u(f(t))$; and neutrality if $u(f(t), p^*) = u(f(t))$. Consequently, if Assumption 2 holds, there is local uncertainty aversion if there is a $p^* \in P$ such that

$$\lim_{t \to 0} \frac{d}{dt} \left(u(f(t), p^*)) - u(f(t))\right)$$

is positive, and only if it is nonnegative, for all $x \in X$ and $f \in F$; attraction if it is negative, and only if it is nonpositive; and neutrality if and only if it is equal to zero.

Proposition 26. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits local uncertainty aversion if there is a $p^* \in P$ such that $MEV(x, (f, p^*)) - MEV(x, f)$ is positive (these functions are introduced by Definitions 3 and 6), and only if it is nonnegative, for all $x \in X$, $f \in F_n$ and $n = 1, ..., m$; attraction if $MEV(x, (f, p^*)) - MEV(x, f)$ is negative and only if it is nonpositive; and neutrality if and only if $MEV(x, (f, p^*)) = MEV(x, f)$.

Proof: Analogous to the proof of Proposition 4. □

Proposition 27. If Assumption 2 holds, $\langle F, \gtrsim \rangle$ exhibits global uncertainty aversion if there is a $p^* \in P$ such that $MEV(x, (f, p^*)) \geqslant MEV(x, f)$ for all $x \in X, f \in F_n$ and $n = 1, ..., m$, and there are two utility functions $\hat{u}_q : (F \times \{p^*\}) \to \mathbb{R}$ and $\hat{u}_f : F \to \mathbb{R}$ such that $\hat{u}_q(x) = \hat{u}_f(x)$ for all $x \in X$ and $\hat{u}_f(f) - \hat{u}_q(f, p^*)$ is a concave function of $(x_i)_{i=1}^n$; and attraction if $MEV(x, (f, p^*)) \leqslant MEV(x, f)$ and $\hat{u}_f(f) - \hat{u}_q(f, p^*)$ is convex.

Proof: Analogous to the proof of Proposition 5, taking into consideration the function $UP_u(t;\ f,\ p^*) = \hat{u}_q(f(t), p^*) - \hat{u}_f(f(t))$, where $f(t) = (x_i(t), e_i)_{i=1}^n$ and $x_i(t) = tx_i + (1 - t)x$. □

Definition 26. (Local risk aversion): $\langle F, \succsim \rangle$ exhibits local risk aversion if for every $x \in X, f \in F$ and $p \in P$ there is a $t^* > 0$ such that $u(E(f(t), p)) \geqslant u(f(t), p)$ for every $t \in [0, t^*]$, where $(f(t), p) = (x_i(t), p(e_i))_{i=1}^n$ with $x_i(t) = x + t(x_i - x)$ for $i = 1, ..., n$; attraction if $u(E(f(t), p)) \leqslant u(f(t), p)$; and neutrality if $u(E(f(t), p)) = u(f(t), p)$.

Remark: This definition coincides with Definition 18, which has been introduced with reference to a DMUR situation. This implies the following Propositions 28 and 29, which coincide with Propositions 4 and 5.

Proposition 28. If Assumption 2 holds, $\langle F, \succsim \rangle$ exhibits local risk aversion if $EV(x, f, p) - MEV(x, (f, p))$ is positive, and only if it is nonnegative, for all $x \in X, f \in F_n$, $n = 1, ..., m$ and $p \in P$; attraction if $EV(x, f, p) - MEV(x, (f, p))$ in negative, and only if it is nonpositive; and neutrality if and only if $EV(x, f, p) = MEV(x, (f, p))$.

Proposition 29. If Assumption 2 holds, $\langle F, \succsim \rangle$ exhibits global risk aversion if $EV(E(f, p), f, p) \geqslant MEV(E(f, p), (f, p))$ for all $f \in F_n$, $n = 1, ..., m$ and $p \in P$, and $u : F \times P \to \mathbb{R}$ is a concave function of $(x_i)_{i=1}^n$; risk attraction if $EV(E(f, p), f, p) \leqslant MEV(E(f, p), (f, p))$ and u is convex; and risk neutrality if $EV(E(f, p), f, p) = MEV(E(f, p), (f, p))$ and u is linear.

4.4. Aversion to increasing uncertainty and to PM-increasing risk

Proposition 30. $\langle F, \succsim \rangle$ exhibits aversion to increasing uncertainty & PM-decreasing risk (introduced by Definition 22) if and only if the set $\bar{G}_F^C(u)$ (introduced by Definition 11) is convex with respet to probabilistic mixtures for all $u \in \mathbb{R}$, i.e., if and only if $f_a, f_b \in \bar{G}_F^C(u)$ implies $(\lambda f_a \oplus (1 - \lambda) f_b) \in \bar{G}_F^C(u)$ for all $\lambda \in [0, 1]$; attraction if $G_F(u)$ is convex; and neutrality if and only if $G_F(u)$ and $\bar{G}_F^C(u)$ are convex.

Proof: Analogous to the proof of Proposition 9. □

Proposition 31. $\langle F, \succsim \rangle$ exhibits aversion to PM-increasing risk (introduced by Definition 23) if and only if for every $p \in P$ the set $G_Q(u)$ (introduced by Definition 9, taking into account that $F \times P = Q$) is convex with respect to probabilistic mixtures for all $x \in X$, i.e., if and only if $(f_a, p), (f_b, p) \in G_Q(u)$ implies $(\lambda f_a \oplus (1 - \lambda) f_b, p) \in G_Q(u)$ for all $\lambda \in [0, 1]$.

Remark: This Proposition coincides with Proposition 9.

4.5. Aversion to risk and uncertainty and to increasing risk and uncertainty when the Savage Expected Utility model applies

If the Savage Expected Utility model applies, i.e., if there are a utility function $U : X \to \mathbb{R}$ and a probability function $p : 2^S \to [0, 1]$ (with $p(\varnothing) = 0, p(S) = 1, p(e) \leqslant p(e')$ for every pair $e, e' \in 2^S$ with $e \subseteq e'$, and $p(e_i) + p(e_j) = p(e_i \cup e_j) + p(e_i \cap e_j)$ for every pair $e_i, e_j \in 2^S$), such that the acts $f \in F$ are ordered according to the SEU function $SEU : F \to \mathbb{R}$, with $SEU(f) = \Sigma_{i=1}^n p(e_i)U(x_i)$, where $f = (x_i, e_i)_{i=1}^n$, then $\langle F, \succsim \rangle$ exhibits uncertainty neutrality (introduced by Definition 20) so that the following propositions hold.

Proposition 32. When the SEU model applies, $\langle F, \succsim \rangle$ exhibits risk aversion if and only if the utility function $U : X \to \mathbb{R}$ is concave, attraction if and only if U is convex; and neutrality if and only if it is linear.

Proposition 33. When the SEU model applies, then $\langle F, \succsim \rangle$ exhibits neutrality to increasing uncertainty & PM-decreasing risk (introduced by Definition 22), PM-increasing risk (introduced by Definition 23) and increasing uncertainty (according to the second remark to Definition 23).

4.6. Aversion to risk and uncertainty and to increasing risk and uncertainty when the Choquet Expected Utility model applies

If the Choquet Expected Utility model (introduced by Schmeidler, 1989, and Gilboa, 1987) applies, i.e., if there are a utility function $U : X \to \mathbb{R}$ and a capacity function $v : 2^S \to [0, 1]$ (with $v(\varnothing) = 0$, $v(S) = 1$, $v(e) \leqslant v(e')$ for every pair $e, e' \in 2^S$ with $e \subseteq e'$), such that the acts $f \in F$ are ordered according to the CEU function $CEU : F \to \mathbb{R}$, with $CEU(f) = U(x_n) + \Sigma_{i=1}^{n-1}(U(x_i) - U(x_{i+1}))\, v(\bigcup_{v=1}^{i} e_v)$, where $f = (x_i, e_i)_{i=1}^n$, with $U(x_i) \geqslant U(x_{i+1})$ for $i = 1, ..., n - 1$, and the lotteries $(f, p) \in F \times P$ induced through the acts by some probability are orderd according to the EU function $EU : F \times P \to \mathbb{R}$, with $EU(f, p) = \Sigma_{i=1}^n p_i U(x_i)$ then the following propositions hold.

Proposition 34. When the CEU model applies, then $\langle F, \succsim \rangle$ exhibits risk aversion (introduced by Definition 21) if and only if the utility function $U : X \to \mathbb{R}$ is concave; attraction if and only if U is convex; and neutrality if and only if it is linear.

Proposition 35. (Montesano and Giovannoni, 1996, and Ghirardato and Marinacci, 1998). When the CEU model applies, then $\langle F, \succsim \rangle$ exhibits uncertainty aversion (introduced by Definition 20) if and only if $\text{core}(v) \neq \varnothing$; attraction if and only if $\text{core}(\bar{v}) \neq \varnothing$; and neutrality if and only if the capacity v is a probability, where $\text{core}(v) = \{p \in P : p(e) \geqslant v(e) \text{ for every } e \in 2^S\}$ and $\text{core}(\bar{v}) = \{p \in P : p(e) \leqslant v(e) \text{ for every } e \in 2^S\}$.

Proposition 36. When the CEU model applies, then $\langle F, \succsim \rangle$ exhibits risk & uncertainty aversion (introduced by Definition 19) if U is concave and $\text{core}(v) \neq \varnothing$; attraction if U is convex and $\text{core}(\bar{v}) \neq \varnothing$; neutrality if and only if U is linear and v is a probability.

Proposition 37. When the CEU model applies, then $\langle F, \gtrsim \rangle$ exhibits neutrality to PM-increasing risk (introduced by Definition 23); he/she exhibits aversion to increasing uncertainty (introduced by Definition 22 and the second remark to Definition 23) if and only if v is convex; attraction if and only if v is concave; neutrality if and only if v is a probability, where v is convex if $v(e_i) + v(e_j) - v(e_i \cup e_j) - v(e_i \cap e_j)$ is nonpositive for every pair $e_i, e_j \in 2^S$ and v is concave if it is nonnegative.

Proof: This proposition was introduced by Schmeidler (1989), who called «uncertainty aversion» what here is indicated as «aversion to increasing uncertainty», and by Montesano and Giovannoni (1996). A simple demostration is the following. Let $\pi = \begin{Bmatrix} 1, & ..., & n \\ \pi_1, & ...\pi_n \end{Bmatrix}$ be a permutation of $\{1, ..., n\}$ and π^1 be a permutation with only one inversion, i.e., $\pi^1_i = i$ for $i = 1, ..., n$ bar a pair π^1_r, π^1_{r+1} with $\pi^1_r = r + 1$ and $\pi^1_{r+1} = r$. Let $CEU\ (f; \pi) = U(x_{\pi_n}) + \Sigma_{i=1}^{n-1} \left(U(x_{\pi_i}) - U(x_{\pi_{i+1}})\right) v(\bigcup_{\nu=1}^{i} e_{\pi_\nu})$. We find that $CEU(f; \pi^1) - CEU(f) = \left(U(x_r) - U(x_{r+1})\right) \left(v(\bigcup_{\nu=1}^{r-1} e_\nu) + v(\bigcup_{\nu=1}^{r+1} e_\nu) - v(\bigcup_{\nu=1}^{r} e_\nu) - v(\bigcup_{\nu=1}^{r-1} e_\nu \cup e_{r+1})\right)$, which is nonnegative (nonpositive) if v is convex (concave). Since every permutation π can be obtained through a sequence of permutations with only one inversion, we have that $CEU(f; \pi) - CEU(f) \geqslant 0$ if v is convex. Since $CEU\ (\lambda f_a \oplus (1 - \lambda) f_b) = \lambda U(x^a_{\pi_n}) + (1 - \lambda)\ U(x^b_{\pi_n}) + \Sigma_{i=1}^{n-1} \left(\lambda U(x^a_{\pi_i}) + (1 - \lambda)\ U(x^b_{\pi_i}) - \lambda U(x^a_{\pi_{i+1}}) - (1 - \lambda)\ U(x^b_{\pi_{i+1}})\right) v(\bigcup_{\nu=1}^{i} e_{\pi_\nu})$ (where π is a permutation of $\{1, ..., n\}$ such that $\lambda U(x^a_{\pi_i}) + (1 - \lambda)\ U(x^b_{\pi_i}) \geqslant \lambda U(x^a_{\pi_{i+1}}) + (1 - \lambda)\ U(x^b_{\pi_{i+1}}))$ for $i = 1, ..., n - 1$ and $e_i = \{s_j \in S : x^a(s_j) = x^a_i$ and $x^b(s_j) = x^b_i\}$ for $i = 1, ..., n$), then $CEU\ (\lambda f_a \oplus (1 - \lambda) f_b) = \lambda CEU(fa; \pi) + (1 - \lambda)\ CEU(f_b; \pi) \geqslant \lambda CEU(f_a) + (1 - \lambda)\ CEU(f_b) \geqslant \min\{CEU(f_a), CEU(f_b)\}$ if v is convex. Therefore, if v is convex, then $\langle F, \gtrsim \rangle$ exhibits uncertainty aversion. If v is not convex, then there is a pair $e_i, e_j \in 2^S$ such that $v(e_i) + (e_j) > v(e_i \cup e_j) + v(e_i \cap e_j)$. Let the following two acts be taken into consideration: $f_a = (x_1, e_i; x_4, S \backslash e_i)$ and $f_b = (x_2, e_j; x_3, S \backslash e_j)$, with $U(x_1) \geqslant U(x_2) > U(x_3) \geqslant U(x_4)$ and $\left(U(x_1) - U(x_4)\right) v(e_i) = U(x_3) - U(x_4) + \left(U(x_2) - U(x_3)\right) v(e_j)$ so that $CEU(f_a) = CEU(f_b)$. Let $\lambda = \dfrac{U(x_2) - U(x_3)}{U(x_1) + U(x_2) - U(x_3) - U(x_4)}$, so that $\lambda U(x_4) + (1 - \lambda) U(x_2) = \lambda U(x_1) + (1 - \lambda) U(x_3)$. Then, $CEU\left(\lambda f_a \oplus (1 - \lambda) f_b\right) = \lambda U(x_4) + (1 - \lambda) U(x_3) + \lambda\left(U(x_1) - U(x_4)\right) v(e_i \cap e_j) + \lambda\left(U(x_1) - U(x_4)\right) v(e_i \cup e_j) = \lambda CEU f_a) + (1 - \lambda)\ CEU(f_b) - \lambda\left(U(x_1) - U(x_4)\right) \left(v(e_i) + v(e_j) - v(e_i \cup e_j) - v(e_i \cap e_j)\right) < \lambda CEU(f_a) + (1 - \lambda)\ CEU(f_b) = \min\{CEU(f_a), CEU(f_b)\}$ since $CEU(f_a) = CEU(f_b)$. Consequently, if v is not convex, then it is not $CEU(\lambda f_a \oplus (1 - \lambda) f_b) \geqslant \min\{CEU(f_a), CEU(f_b)\}$ for every pair $f_a, f_b \in F$ and every $\lambda \in [0, 1]$. □

5. CONCLUSIONS

This paper supplies the extension to the multidimensional case of many propositions previously introduced (Montesano, 1999a and b) for the unidimensional case. Since the unidimensional case is also included in the multidimensional case, the propositions found in Sections 3 and 4 are quite similar to the corresponding unidimensional propositions. Nevertheless, there are some important differences, which can be summarized as follows.

First, the comparative analysis is not very significant (Proposition 1). Second, the extension of the mean-preserving-spreads increasing risk has been introduced (Definition 8) and it leads to Propositions 7 and 8, which are very similar to those found in the unidimensional case. Third, the propositions which require the expected values of lotteries and acts (Propositions 3, 4, 5, 21, 22, 23, 24, 25, 26, 27, 28 and 29) imply that values be associated to the k commodities (or dimensions) taken into account by the set of consequences. These values have been introduced (Definition 2) in an endogenous way through a kind of marginal rates of substitution. Finally, we can note that not all the unidimensional propositions introduced by the proceding papers can be extended to the multidimensional case. However they suprisingly result very few.

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